SIMPLICES AND SIMPLICIAL DECOMPOSITIONS

Barycentric coordinates. In the drawing below, each of the points **P, Q, R** lies in the plane determined by $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, and consequently each can be written as a linear combination $w_1\mathbf{P}_1 + w_2\mathbf{P}_2 + w_3\mathbf{P}_3$, where $w_1 + w_2 + w_3 = 1$. For the point **P**, the barycentric coordinates w_i are all positive, while for the point **R** the barycentric coordinates are such that $w_1 = 0$ but the other two are positive, and for the point **Q** the barycentric coordinates are such that w_1 is negative but the other two are positive.

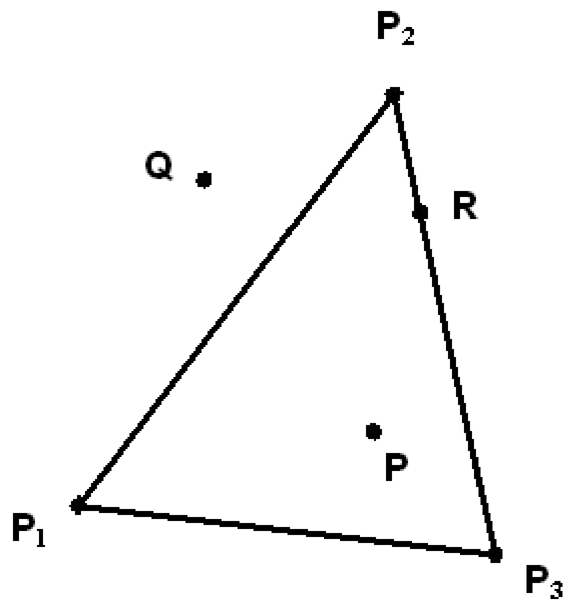


FIGURE 1

(***Source:*** http://graphics.idav.ucdavis.edu/education/GraphicsNotes/Barycentric-Coordinates/Barycentric-Coordinates.html)

Examples of points for which w_2 is positive but the remaining coordinates are negative can also be constructed using this picture; for example, if one takes the midpoint **M** of the segment $[\mathbf{P}_1\mathbf{P}_3]$, then the point $\mathbf{S} = 2\mathbf{P}_2 - \mathbf{M}$ will have this property (geometrically, $\mathbf{P}_2$ is the midpoint of the segment joining **M** and **S**).

Illustration of a 2 – simplex . We shall use a modified version of Figure 1; the points of the 2 – simplex with vertices $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ consists of the triangle determined by these points and the points which lie inside this triangle (in the usual intuitive sense of the word).

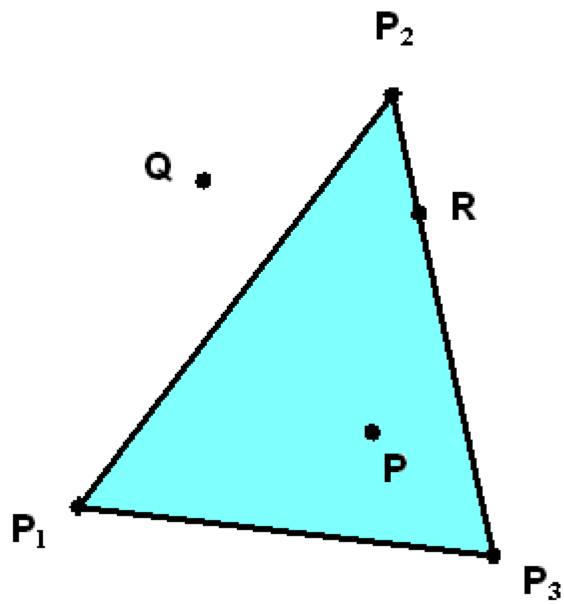
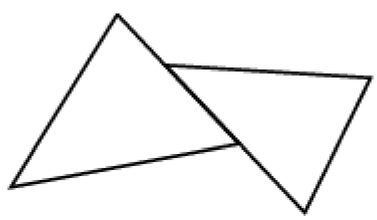


FIGURE 2

In this picture the points **P** and **R** lie on the simplex $\mathbf{P_1P_2P_3}$ because their barycentric coordinates are all nonnegative, but the point **Q** does not because one of its barycentric coordinates is negative.

Note that the ***(proper) faces*** of this simplex are the closed segments $\mathbf{P_1P_2}$, $\mathbf{P_2P_3}$, and $\mathbf{P_1P_3}$ joining pairs of vertices as well as the three vertices themselves (and possibly the empty set if we want to talk about an empty face with no vertices).

Simplicial decompositions. It is useful to look at a few spaces given as unions of $2-$ simplices, some of which determine simplicial complexes in the sense of the notes and others that do not.



not a simplicial complex

FIGURE 3

(***Source:*** **http://mathworld.wolfram.com/SimplicialComplex.html**)

In the example above the intersection of the $2-$ simplices is not a common face. On the other hand, we can split the two simplices into smaller pieces such that we do have a simplicial decomposition.

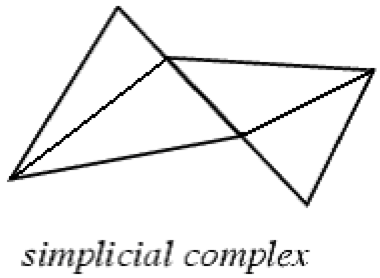
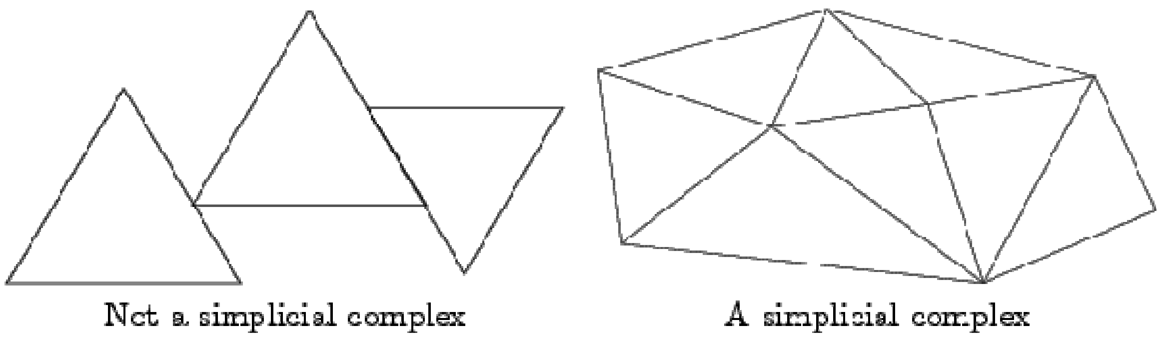


FIGURE 4

Here are two more examples; in the second case the simplices determine a simplicial complex and in the first they do not. As in the preceding example, one can subdivide the simplices in the first example to obtain a simplicial decomposition.



(***Source:*** **http://planning.cs.uiuc.edu/node274.html**)

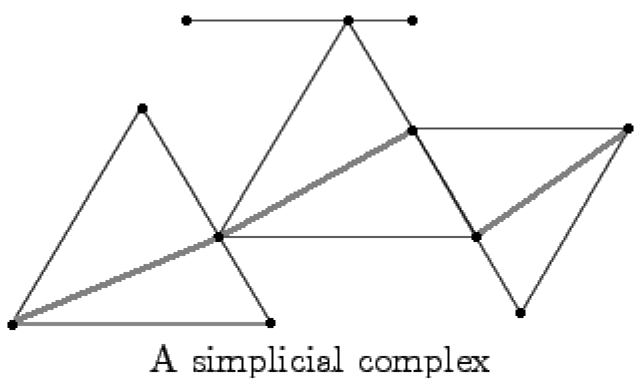


FIGURE 5

Triangulations. In the example from page **523** of Marsden and Tromba, the annulus bounded by two circles is split into four isometric pieces as in the drawing on the next page.

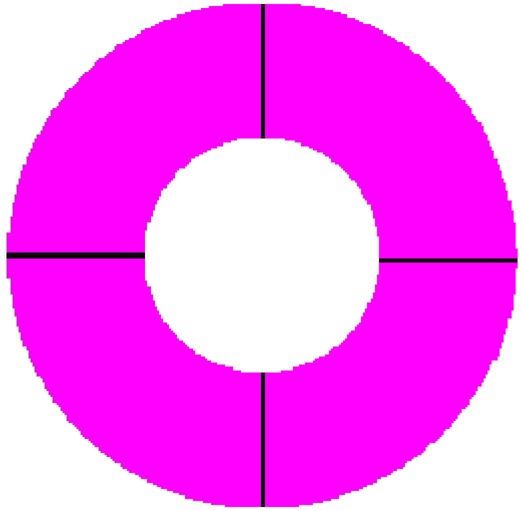


FIGURE 6

Each of the four pieces is homeomorphic to a solid rectangle**.** Since a solid rectangle has a simplicial decomposition into two **2 –** simplices**,** one can use such a decomposition to form a triangulation of the solid annulus**.**

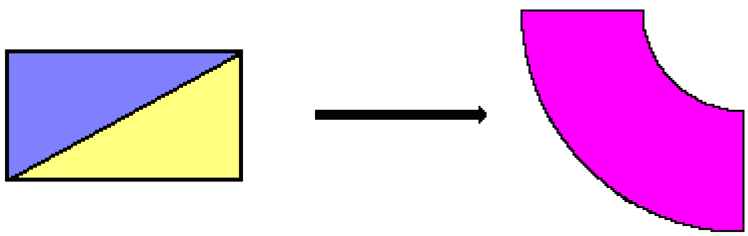


FIGURE 7

A closely related way of triangulating the annulus is suggested by the figure below**:**

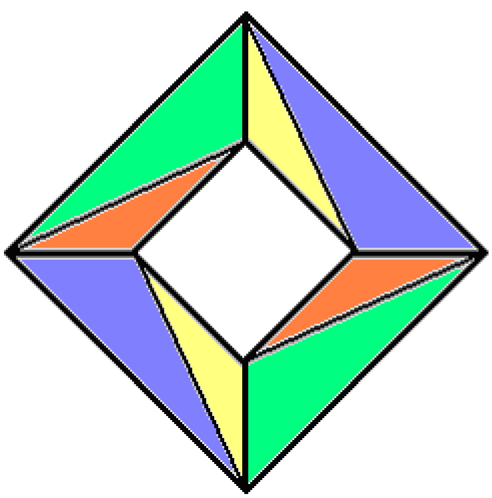


FIGURE 8

Similarly, many familiar closed polygonal regions can be triangulated fairly easily. Here is an example for a solid hexagon.

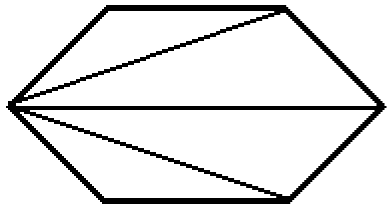


FIGURE 9

And here is a more complicated example of a closed nonconvex polygonal region which can easily be triangulated.

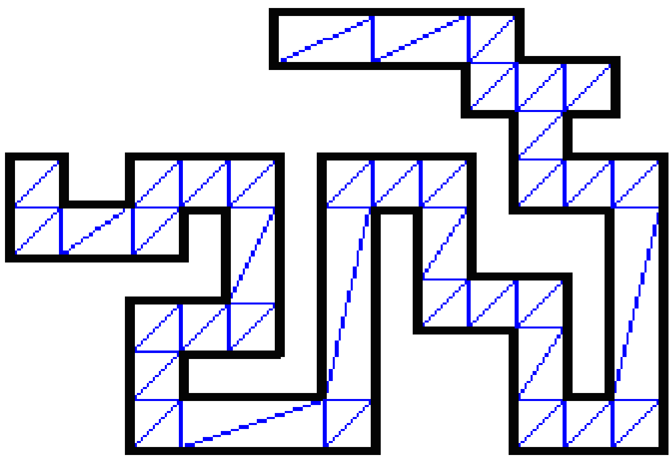
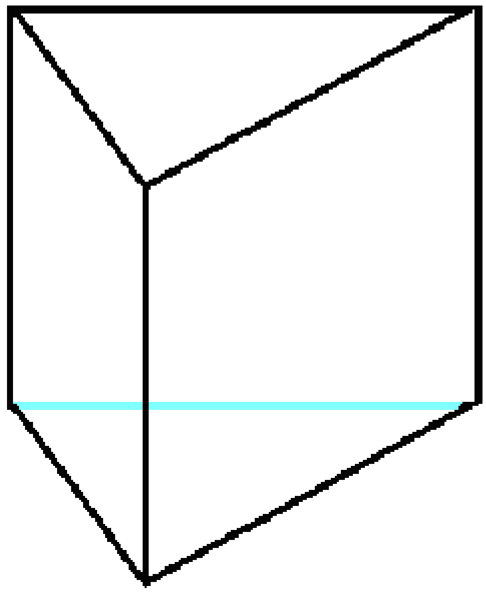
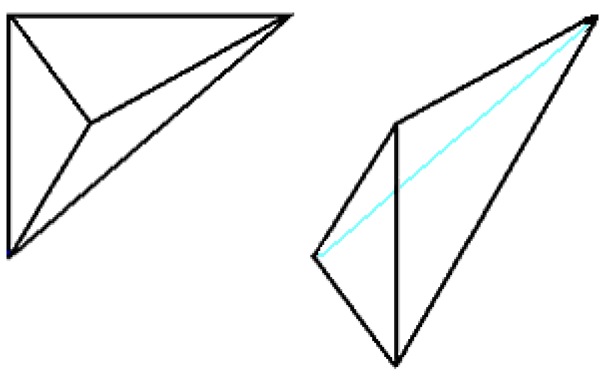


FIGURE 10

Triangulations of prisms. The drawings below illustrate the standard decomposition of a **3 –** dimensional triangular prism**.**



If we take $\mathbf{x_0}$, $\mathbf{x_1}$, and $\mathbf{x_2}$ to be the vertices of the bottom triangle and $\mathbf{y_0}$, $\mathbf{y_1}$, and $\mathbf{y_2}$ to be the vertices of the top triangle, then the decomposition is given as follows:



$\mathbf{x_0 y_0 y_1 y_2}$ $\mathbf{x_0 x_1 y_1 y_2}$ $\mathbf{x_0 x_1 x_2 y_2}$

Since this decomposition may be difficult to visualize, there is another illustration of this decomposition in the file **http://math.ucr.edu/~res/math205B-2012/prism-dissection.pdf**.