

Mathematics 205C, Spring 2011, Final Examination

Solutions

1. [25 points] Let (X, \mathcal{E}) be a connected graph, let $n \geq 2$ be an integer, and let (X', \mathcal{E}') be a connected n -sheeted covering with the canonical graph structure. Suppose that $\pi_1(X)$ is a free group on $g \geq 1$ generators. Then $\pi_1(X')$ is a free group on $h \geq 1$ generators for some positive integer h . Derive a formula for h in terms of g and n . [Hint: Look at the Euler characteristics of X and X' .]

SOLUTION

The Euler characteristics of X and X' are given by $\chi(X) = 1 - g$ and $\chi(X') = 1 - h$. Since X' is an n -sheeted covering of X we have $\chi(X') = n\chi(X)$. Therefore

$$h = 1 - \chi(X') = 1 - n\chi(X) = 1 - n(1 - g) = ng - n + 1.$$

2. [25 points] Let A_* and B_* be chain complexes, and suppose that $f : A_* \rightarrow B_*$ is a map of chain complexes. Prove that for each integer q there is a well-defined homomorphism of homology groups $f_* : H_q(A) \rightarrow H_q(B)$ such that if $u \in H_q(A)$ is represented by $x \in A_q$, then $f_*(u)$ is represented by $f(x) \in B_q$.

SOLUTION

First of all we need to prove that if $dx = 0$ then $df(x) = 0$; but $dx = 0$ and the fact that f is a chain map imply

$$0 = f(dx) = df(x)$$

so this is true. Next, suppose that x and y represent u , so that $x - y = dz$ for some z . Then

$$f(y) = f(x - dz) = f(x) - f(dz) = f(x) - df(z)$$

so that $f(x)$ and $f(y)$ represent the same element in $H_q(B)$. To see the map is a homomorphism, note that if x_i represents u_i for $i = 1, 2$, then $f_*(u_1 + u_2)$ is represented by

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

and by the basic condition in the problem the right hand side represents $f_*(u_1) + f_*(u_2)$.

3. [25 points] If $k \geq 1$ then the half-space \mathbb{R}_+^k is defined to be the set of all points in \mathbb{R}^k whose first coordinates are nonnegative. Prove that if $m \neq n$ are positive integers then \mathbb{R}_+^m and \mathbb{R}_+^n are not homeomorphic. [Hint: \mathbb{R}_+^k has an open dense subset which is open in \mathbb{R}^k ; namely, the set of all points whose first coordinates are positive. If X is homeomorphic to both of these spaces, why does X have open dense sets homeomorphic to \mathbb{R}^m and \mathbb{R}^n ? Recall that the intersection of two such open dense subsets is always dense and hence nonempty.]

SOLUTION

Suppose there is a homeomorphism $f : \mathbb{R}_+^m \rightarrow \mathbb{R}_+^n$. Let $V \subset \mathbb{R}_+^n$ be the open subset (even in \mathbb{R}^n) of all points whose first coordinates are positive, let $U_0 \subset \mathbb{R}_+^m$ be the open subset (even in \mathbb{R}^m) of all points whose first coordinates are positive, and let $U = f[U_0]$. Since f is a homeomorphism we have $U = \mathbb{R}_+^n - [\mathbb{R}^{n-1} \times \{0\}]$, and since f is a homeomorphism the deleted set is closed in \mathbb{R}_+^n , so that U is also open in \mathbb{R}_+^n . Clearly U and V are both dense subsets (every point in \mathbb{R}_+^k is a limit of points whose last coordinates are positive), and since the intersection of two open dense subsets is open and dense we conclude that $U \cap V$ is also dense in \mathbb{R}_+^n . Now $U \cap V$ is open in both U and V , where one of the latter is homeomorphic to an open subset of \mathbb{R}^m and the other is homeomorphic to an open subset of \mathbb{R}^n . By Invariance of Dimension, this implies that $m = n$.

4. [25 points] (a) Let Δ_3 be the standard 3-dimensional simplex with vertices $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Write down the explicit formula for the boundary of the generator $\mathbf{e}_0 \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$ in the simplicial chain complex $C_*(\Delta_3)$ as a linear combination of the generators for $C_2(\Delta_3)$.

(b) Suppose that (P, \mathbf{K}) is a connected simplicial complex, and let $A, B \in \mathbf{K}$. Prove that there is a sequence of simplices $A = S_0, \dots, S_p = B$ such that each intersection is nonempty. [Hint: Define a binary relation on simplices such that A and B are related if a chain of the given type exists. Why is this an equivalence relation, and why is the union of all simplices in a given equivalence class a closed and open subset of P ?]

SOLUTION

(a) The boundary is given by

$$\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 - \mathbf{e}_0 \mathbf{e}_2 \mathbf{e}_3 + \mathbf{e}_0 \mathbf{e}_1 \mathbf{e}_3 - \mathbf{e}_0 \mathbf{e}_1 \mathbf{e}_2 .$$

(b) The binary relation on simplices is just the equivalence relation generated by the binary relation $A \sim B$ if and only if $A \cap B$ is nonempty. Given a simplex A , let P_A be the set of all points in simplices which belong to the equivalence class of A , so that P_A is the union of all simplices in this class. Clearly P is the union of the sets P_A , while $A \sim B$ implies $P_A = P_B$ and if $P_A \cap P_B \neq \emptyset$, then a point x in this intersection must lie on some simplex C which is in the same equivalence class as A and likewise for B ; in other words the subsets P_A and P_B are either disjoint or identical. Each of these is a finite union of simplices and hence closed, so we have a decomposition of P into finitely many pairwise disjoint closed subsets. Since P is connected there can be only one subset in this decomposition, so $P = P_C$ for every simplex C , and hence for any two simplices we have $A \sim B$.

5. [25 points] (a) Suppose that X is a connected and locally simply connected space such that $\pi_1(X)$ is finite of odd order. Prove that X has no connected 2-sheeted covering spaces.

(b) Suppose that $X = U \cup V$ where U , V and $U \cap V$ are all open and arcwise connected. Prove that $\pi_1(X)$ is finitely generated if both $\pi_1(U)$ and $\pi_1(V)$ are.

SOLUTION

(a) The equivalence classes of connected coverings are in 1–1 correspondence with subgroups of $\pi_1(X)$, and if a covering space corresponds to the subgroup H then the number of sheets is equal to the index of H in $\pi_1(X)$. If the latter has odd order, then by Lagrange's Theorem on cosets the index of H must also be odd. In particular, it cannot be equal to 2.

(b) By the Seifert-van Kampen Theorem we know that $\pi_1(X)$ is generated by the images of $\pi_1(U)$ and $\pi_1(V)$. If the latter have finite generating sets A and B then the union of their images in $\pi_1(X)$ will be a finite generating set for $\pi_1(X)$.

6. [25 points] Let (P, \mathbf{K}) be an n -dimensional simplicial complex. Prove that $H_n(P, \mathbf{K})$ is a finitely generated free abelian group. [Hint: Why is $H_n(P, \mathbf{K})$ isomorphic to the kernel of d_n ?]

SOLUTION

By definition the homology group is isomorphic to the kernel of D_n modulo the image of d_{n+1} . Since the complex is n -dimensional, the domain of the latter is zero and hence this map is zero, and therefore $H_n(P, \mathbf{K})$ is isomorphic to the kernel of d_n . Since this kernel is a subgroup of the finitely generated abelian group $C_n(P, \mathbf{K})$, it follows that the kernel must be a finitely generated free abelian group. In particular, it has no nonzero elements of finite order.

7. [25 points] Given a space X and $x \in X$, the local homology groups of X at x are defined by $H_*(X, X - \{x\})$. Prove that if U is an open neighborhood of x then these groups only depend on U (and its subspace topology). [Note: Recall the default assumption that X is Hausdorff.]

SOLUTION

We know that $X = X - \{x\} \cup U$ and $U - \{x\} = X - \{x\} \cap U$, so by excision the morphism from $H_*(U, U - \{x\})$ to $H_*(X, X - \{x\})$ induced by inclusion will be an isomorphism.

8. [25 points] Suppose that $X = U \cup V$ where U and V open in X . Assume further that the maps $H_q(U \cap V)$ to $H_q(U)$ induced by the inclusion $U \cap V \subset U$ are isomorphisms (in all dimensions) and the maps $H_q(U \cap V)$ to $H_q(V)$ induced by the inclusion $U \cap V \subset V$ are zero (in all dimensions). Prove that the maps $H_q(V)$ to $H_q(X)$ induced by the inclusion $V \subset X$ are also isomorphisms (in all dimensions).

Extra credit. [20 points] Prove the same conclusion holds without the restriction on the maps $H_q(U \cap V)$ to $H_q(V)$. [Hint: If $f : A \rightarrow B$ is a homomorphism of abelian groups and $F : A \rightarrow A \oplus B$ is the graph map $F(a) = (a, f(a))$, show that the quotient of $A \oplus B$ modulo the image of F is isomorphic to B by considering the map $\varphi : A \oplus B \rightarrow B$ with $\varphi(a, b) = b - f(a)$.]

SOLUTION

Consider the Mayer-Vietoris sequence for (U, V) . By assumption the maps from $H_*(U \cap V)$ to $H_*(U) \oplus H_*(V)$ are 1-1 because this is true for their projections onto $H_*(U)$ are in fact isomorphisms. This means that the boundary maps

$$\Delta : H_{q+1}(U \cup V) \rightarrow H_q(U \cap V)$$

are all zero and the maps from $H_*(U) \oplus H_*(V)$ to $H_*(U \cup V)$ are onto. By exactness we know that the kernel of this onto map is the image of the previous map, which is just $H_*(U) \oplus 0$. But this means that the map from $H_*(V)$ to $H_*(U \cup V)$ must be an isomorphism.

EXTRA CREDIT. In this case we still know that the map from $H_*(U \cap V)$ to direct sum $H_*(U) \oplus H_*(V)$ is 1-1 and hence Δ is also zero in this case, so that the Mayer-Vietoris sequence becomes a collection of short exact sequences:

$$0 \rightarrow H_q(U \cap V) \rightarrow H_q(U) \oplus H_q(V) \rightarrow H_q(U \cup V) \rightarrow 0$$

If we identify $H_q(U \cap V)$ with $H_q(U) = A$ (by definition) using the isomorphism assumption and let $H_q(V) = B$, then the short exact sequence imply that $H_q(U \cup V)$ is the quotient of $A \oplus B$ modulo the image C of a homomorphism $A \rightarrow A \oplus B$ sending a to $(a, g(a))$ for some homomorphism $g : A \rightarrow B$. Therefore it suffices to show that the quotient of $A \oplus B$ by such a subgroup is isomorphic to B . The simplest way to do this is to define a homomorphism from $A \oplus B$ to B which is onto and whose kernel is C , and this can be done by defining $h : A \oplus B \rightarrow B$ so that $h(a, b) = b - g(a)$.

An extra page for use if needed