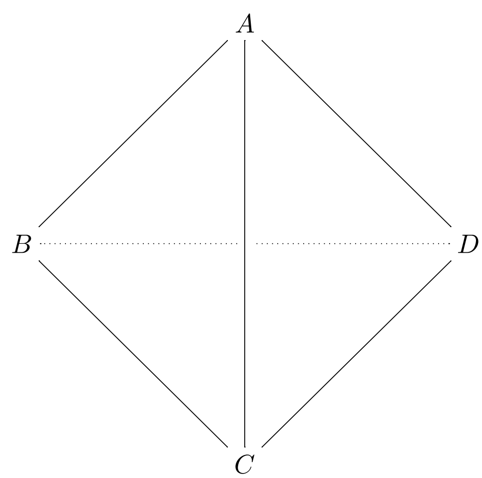
Special Geometry Exam, Fall 2008, W. Stephen Wilson. Mathematics Department, Johns Hopkins University

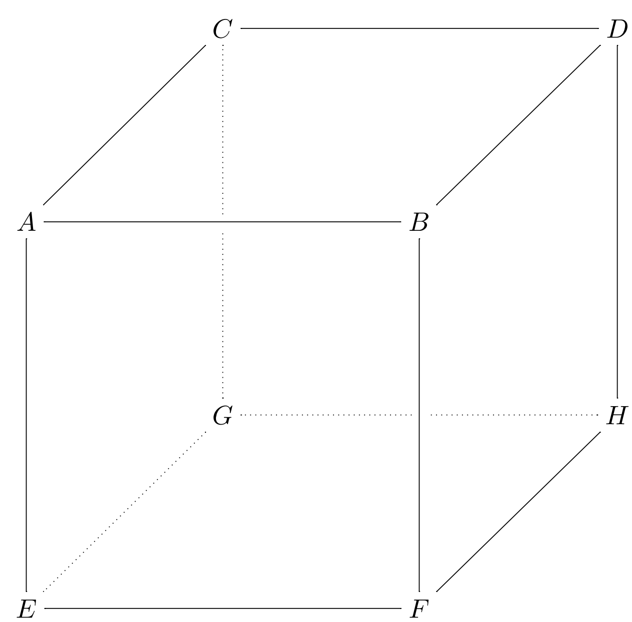
I agree to complete this exam without unauthorized assistance from any person, materials or device.

Name print and sign: ______________________________ Date: ______________

NO CALCULATORS, NO PAPERS, SHOW WORK. Put your final answer in the box provided.

Consider the regular tetrahedron and the cube, both with edges of length one unit.



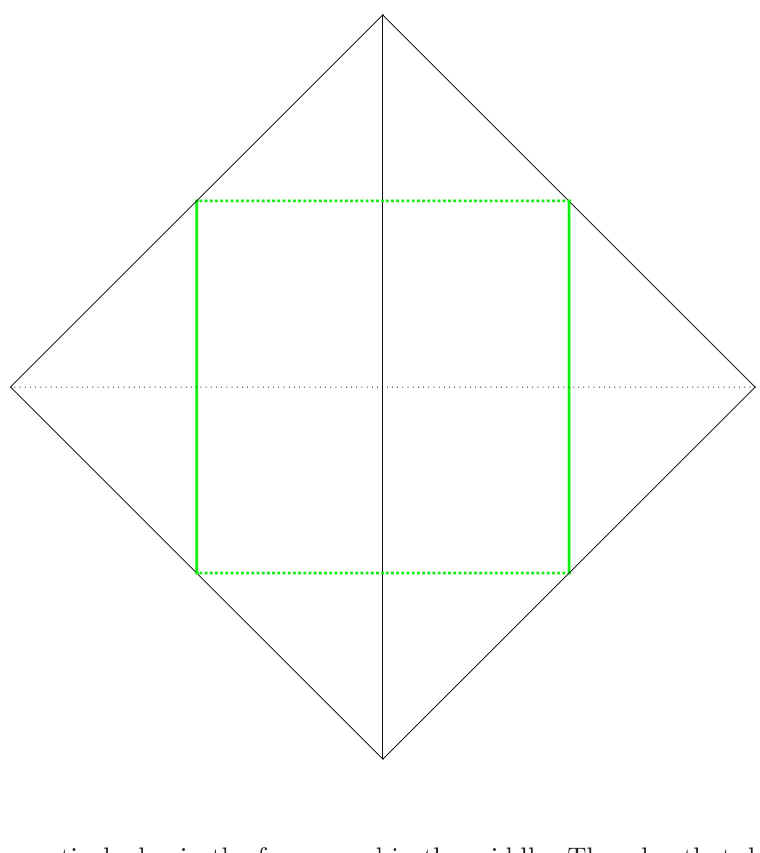


There are three (3) distinct planes that divide the solid regular tetrahedron into two identical pieces and intersect the tetrahedron in a square. Remember, the edges of the tetrahedron are of one unit length.

1. What is the area of the square mentioned above? $\boxed{1/4}$

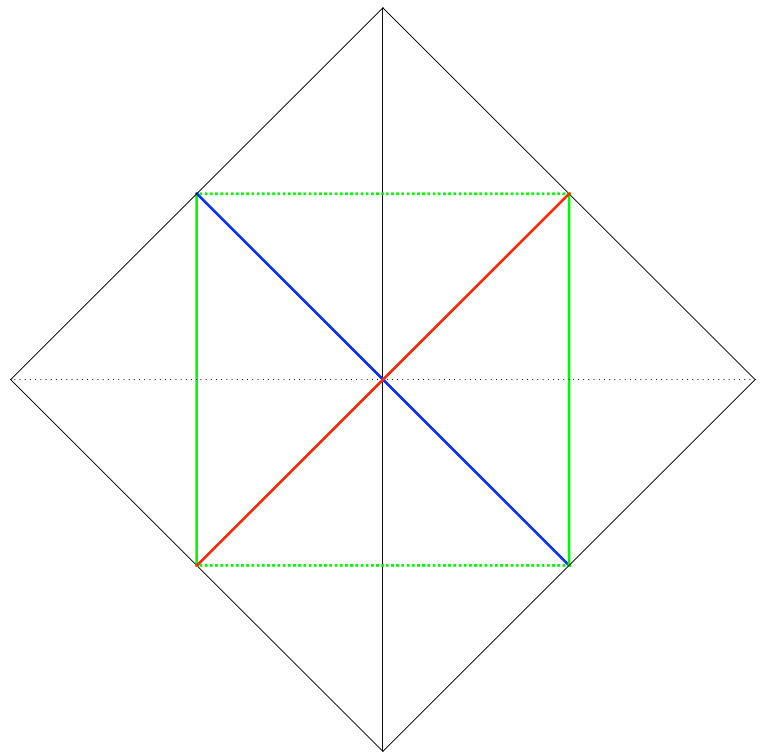
One way to see this square is to start with one edge. There are 6 edges, and 4 of them connect to our chosen edge and there is only one that does not touch it. The four edges that touch our first one also touch our second one. Take the midpoints of those 4 edges. They are the corners of our square. Each edge for the square goes from the midpoint of the side of an equilateral triangle with sides of length one to the midpoint of another side. The length of the side must therefore be 1/2 and the area must be 1/4.

To see a picture, we have:



First, consider the vertical edge in the foreground in the middle. The edge that doesn't touch it is the horizontal edge in the background (dotted). The green lines connect the midpoints of the four other edges to make the square. The two solid green lines are visible and the two dotted green lines are on the other side and so can't be seen.

In our next picture we insert the other two planes. Both the red and the blue show the other two squares on edge. For the red, the two edges that do not touch are the edge at the upper left and the lower right. The red square then uses the midpoints of the other 4 edges. For blue, we use the upper right and the lower left as the two non-touching edges.



A nice exercise is to take a tetrahedron and cut it into two pieces along the square. Smooth them off and offer then to friend or foe and ask them to put the two together to make a tetrahedron. Most people find this very difficult.

2. What is the intersection of two (2) of those planes mentioned above and the tetrahedron? If it is an area, give the area; it if is a line segment, give the length; if it is points, say how many; if there are no points, just put zero (0).

$\boxed{1/\sqrt{2} \text{ length}}$

We have a picture already. Taking the intersection of the green square and the red square we see that we have a line segment with length given by the diagonal of one of the squares, or $\frac{1}{\sqrt{2}}$.

3. What is the intersection of all three (3) of those planes mentioned above and the tetrahedron? If it is an area, give the area; it if is a line segment, give the length; if it is points, say how many; if there are no points, just put zero (0).

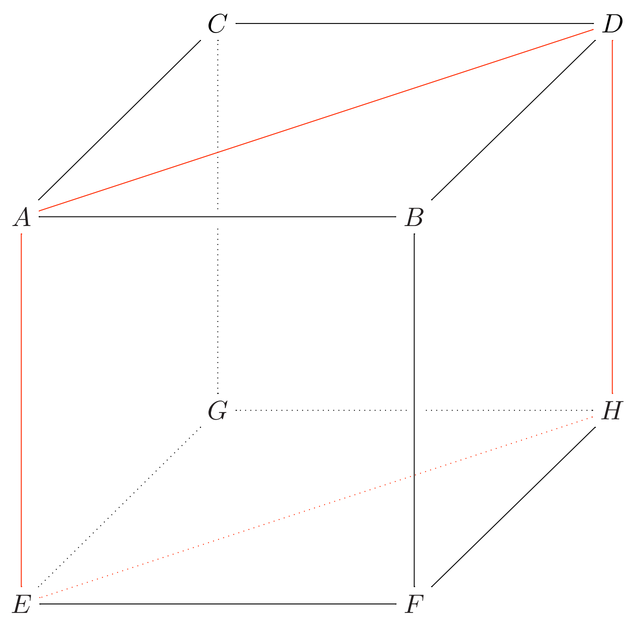
$\boxed{1 \text{ point}}$

Now it is easy to see the three planes and the tetrahedron intersect at one point.

4. Find the area of the rectangle formed by the intersection of a plane with the cube when the plane contains four (4) vertices of the cube but it is not a face of the cube. $\boxed{\sqrt{2}}$

Attempting the few possibilities leaves the plane going through two vertices connected by an edge and two vertices diagonally opposite them, also connected by an edge. An example from the first page would be A, E, D and H. The height is 1 and the length is $\sqrt{2}$ so the area is $\sqrt{2}$.

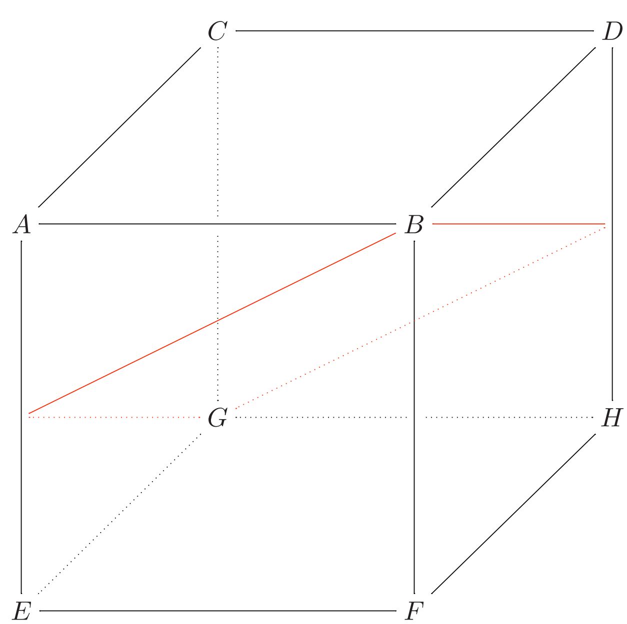
The picture:



5. A rhombus is a parallelogram with all sides the same length. There is a plane that intersects the cube, going through exactly two (2) vertices, such that the intersection is a rhombus. What is the length of the side of the rhombus?

$\boxed{\sqrt{5}/2}$

The picture is



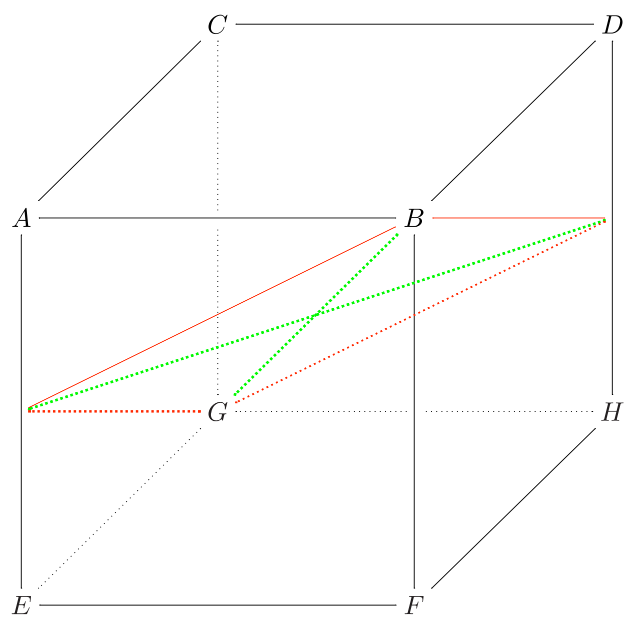
Now it is easy to see that the length of a side is the hypotenuse of a right triangle with sides 1 and 1/2, so we get $\sqrt{5}/2$.

6. What is the height of the above rhombus? $\boxed{\frac{\sqrt{6}}{\sqrt{5}}}$

There is probably some easy, sneaky, way to do this, but I don't see it. My way is to compute the area and then divide by the base length.

The green lines below are the two diagonals of the rhombus. They are perpendicular bisectors and it is easy to compute their lengths. The one from B to G has length $\sqrt{3}$ and the one from the midpoint of AE to the midpoint of DH has length $\sqrt{2}$. If you don't remember the formula for the area of a rhombus you can easily do this from triangles anyway. The area is 1/2 the product of the lengths of the diagonals, or, $\frac{\sqrt{3}}{\sqrt{2}}$.

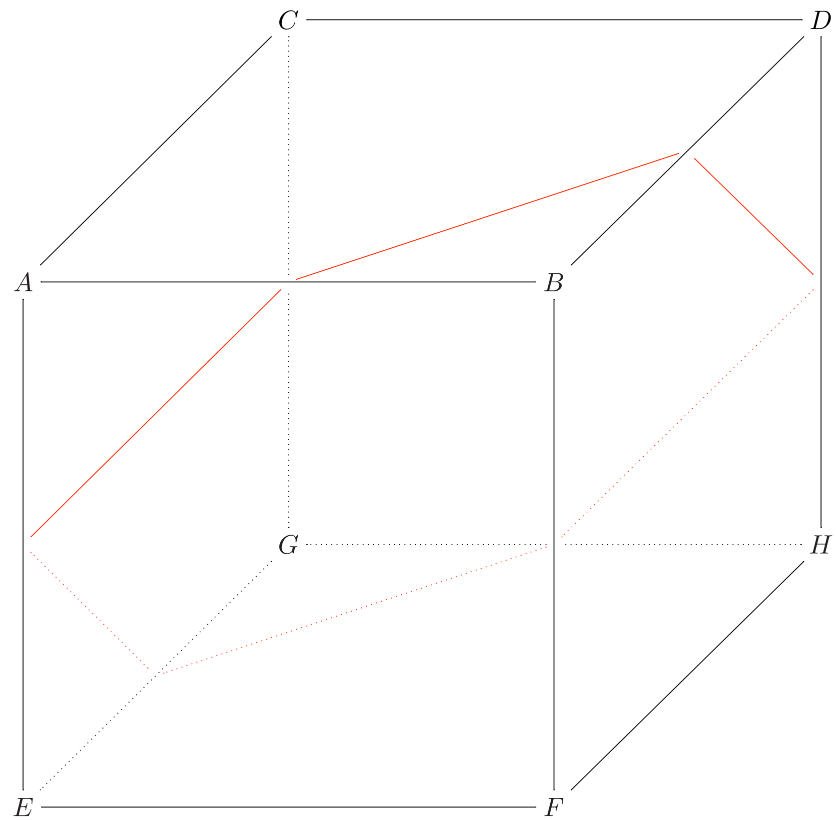
We know the base length is $\frac{\sqrt{5}}{2}$. Divide this into the area to get $\frac{\sqrt{6}}{\sqrt{5}}$.



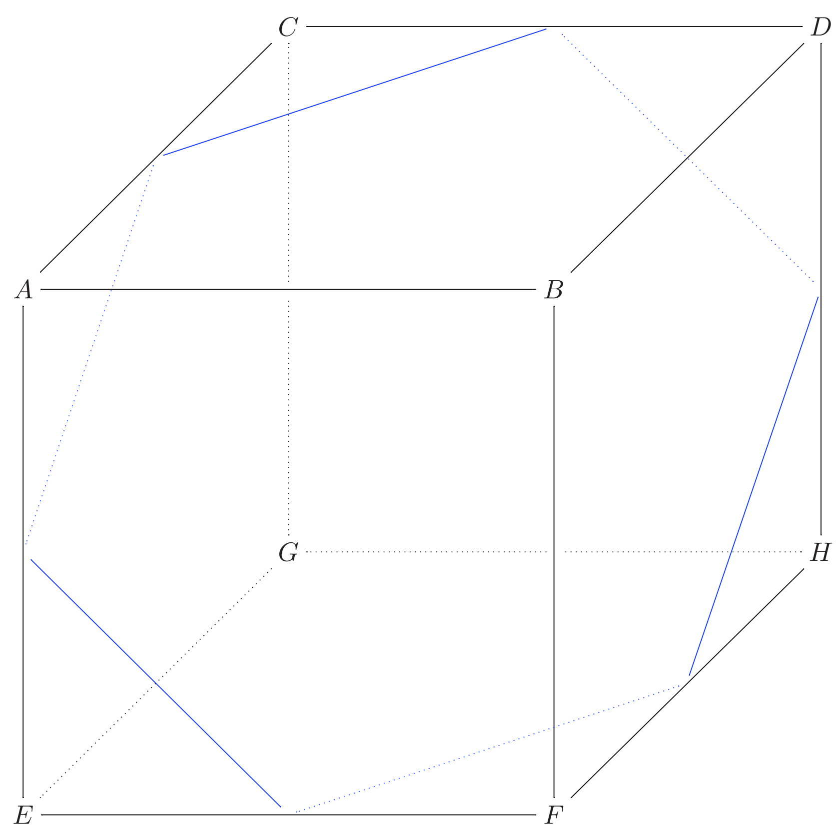
7. There are planes that intersect the cube in a regular hexagon. What is the length of a side of such a hexagon? $\boxed{\frac{1}{\sqrt{2}}}$

Below is one of these hexagons. One way to see them would be to stand the cube on a vertex. In the case of the picture below, stand it like a top with F at the very bottom and C at the top, directly above F. Then take the plane that is half way between C and F and perpendicular to the line between C and F. That plane intersects to cube at the hexagon drawn in red.

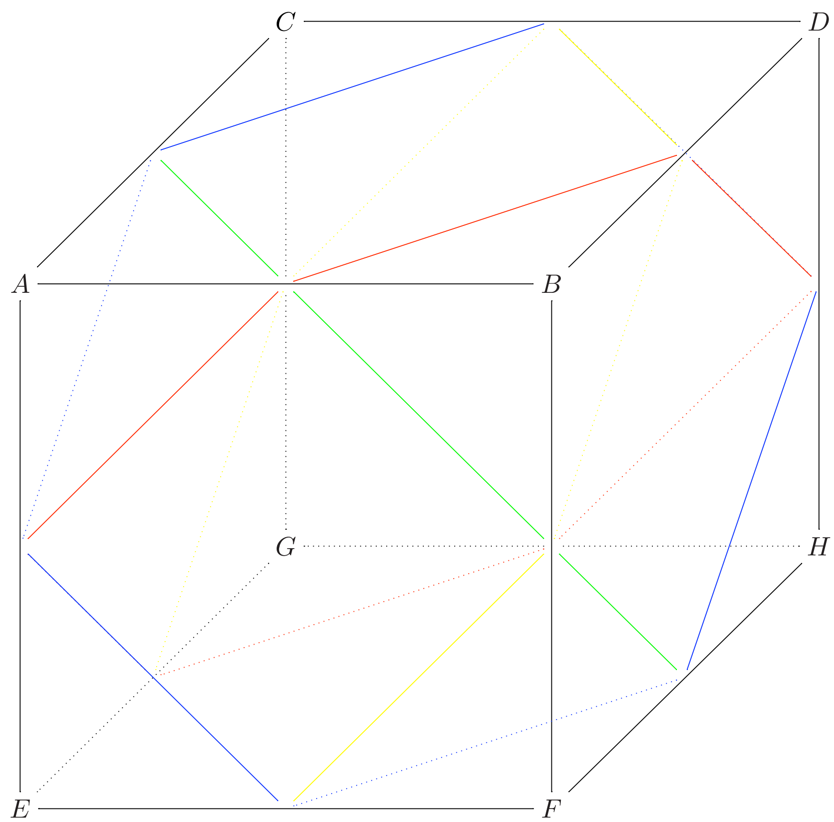
Note that the hexagon must go through all 6 sides of the cube and the vertices are midpoints of edges. The length of a side of the hexagon is now easily calculated using the Pythagorean theorem: $\frac{1}{\sqrt{2}}$.



Here is an alternative version. There are 4 different planes that give regular hexagons and this picture is perhaps the best. This is the plane half way between B and G and perpendicular to the line from B to G.

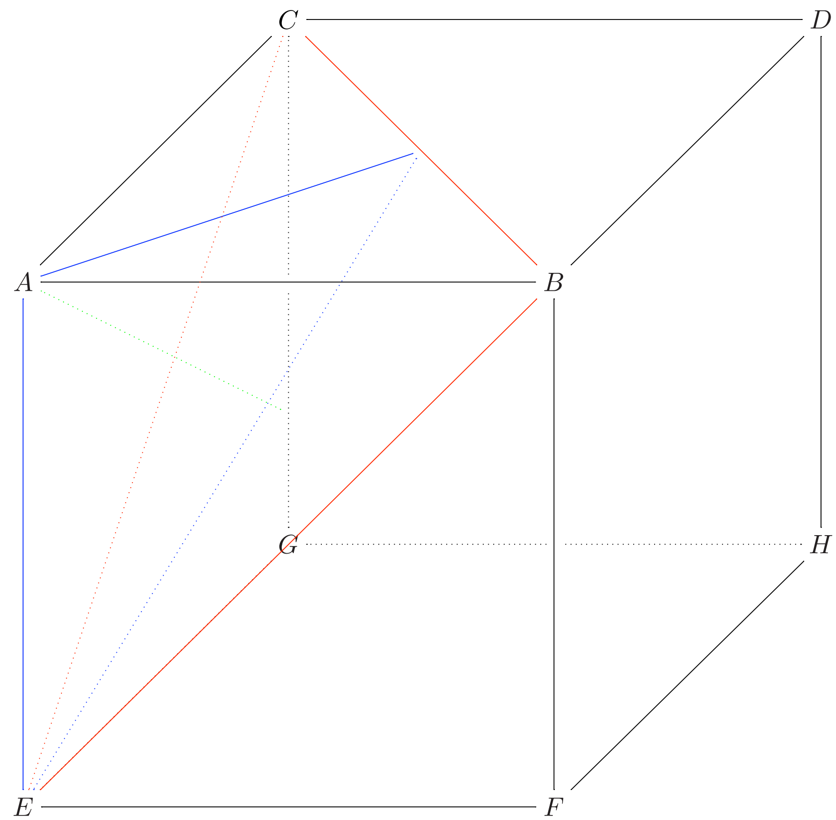


This is messy, but it has all 4 of the hexagons in it. Some of the lines overlap using this view of the cube. This only happens when one is on one side of the cube and the other on the other side.



8. Find a plane that goes through exactly three (3) vertices. On one side of the plane there are four (4) vertices and on the other side there is one (1) vertex. How far is the plane from the single vertex? $\boxed{\frac{1}{\sqrt{3}}}$

I know that there are difficult ways to do this, but my 16 year old son explained from the back of the car how to do it quickly in his head. Consider the diagram:



Here the red lines indicate the intersection of the plane that goes through E, B and C. The blue lines indicate a right triangle made using A, E and the midpoint of the red line between C and B. Drop the (green) perpendicular to the plane formed by the triangle B, C, and E. It is on the line between A and H and it hits the BCE plane on the blue line from E to the midpoint of BC.

The length of the vertical blue line is 1. It is easy to calculate the length of the blue line across the top from A to the midpoint of BC; it is just half of the diagonal from A to D, so it is $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

The blue triangle is a right triangle. The right angle is at A so the area is $\frac{1}{2\sqrt{2}}$ or, $\frac{\sqrt{2}}{4}$.

We can use the Pythagorean theorem to compute the length of the hypotenuse as $\frac{\sqrt{3}}{\sqrt{2}}$.

The green line represents the height of this triangle using the hypotenuse as the base. The length of the green line is also the distance of A from the triangle plane of BCE. Since we know the area of the blue triangle and the length of the (hypotenuse) base, we can compute the height, i.e. the number we want: $2\frac{\sqrt{2}\sqrt{2}}{4\sqrt{3}} = \frac{1}{\sqrt{3}}$.

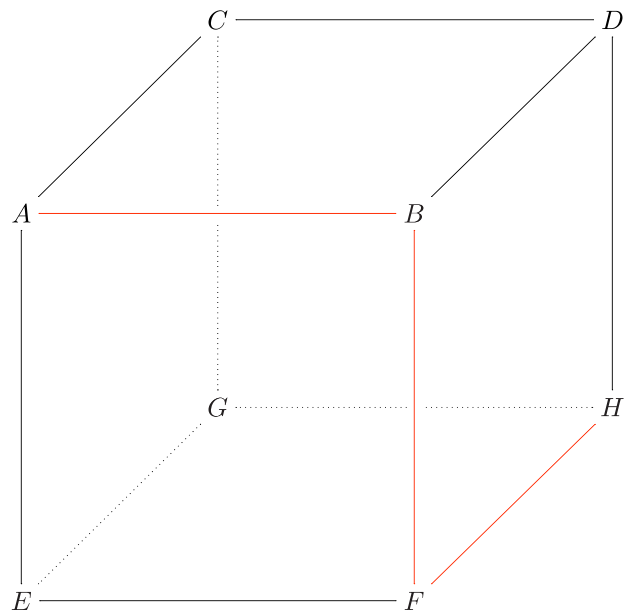
I know there are harder ways to do this because that was how I did it first. (Find an equation for the plane through BCE and the equation for the line through AH. Find their intersection. Compute the distance from that to A.) However, it is also clear that there are easier ways to do this.

The distance between A and H is $\sqrt{3}$ from the Pythagorean theorem applied a couple of times. Note that it turns out that our plane given by BCE is exactly one-third of the way from A to H. (And that the hexagon plane is one-half of the way.)

Knowing the answer suggests that there must be an easy way to see it.

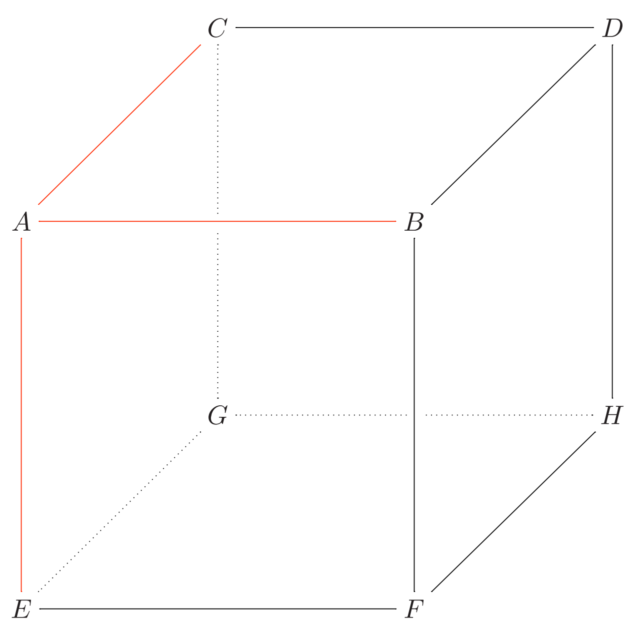
Those who are really smart and/or who have great geometric insight can bypass all of the above and go straight to the correct geometric answer, i.e. one-third of the way between A and H. After thinking about this far too long for me to consider it the "easy" way, I now have the result with no computation, just a little geometric insight.

First, look at the cube again:



We want to show that the line from A to H is divided into 3 equal pieces by the two planes given by BCE and FDG. These lengths correspond to the projections of the three lines AB, BF and FH, since B is in the first plane and F is in the second. As long as we don't change the angle of our lines with respect to the line AH, the projection length will not change. Thus we can move BF over to AE and FH over to AC.

The resulting pictures is:



Now we can compare the projection lengths of AB, AE and AC to the line AH, but since BCE gives the plane we use to divide the line up with, all three of these lines project to the same line segment on AH, i.e. the distance from A to the plane given by BCE. Since all are equal, they must all be exactly one-third the distance from A to H.

9. The plane in the last problem divides the cube into a big piece and a little piece. What is the volume of the big piece?

$\boxed{\frac{5}{6}}$

It is easier to work with the little piece since it is a tetrahedron and we already know its height. The base is just the area of the triangle BCE. BCE has base $\sqrt{2}$ and we have, in the previous problem, computed the height as $\frac{\sqrt{3}}{\sqrt{2}}$. Thus the area of the triangle, i.e. the base of our (non-regular) tetrahedron, is $\frac{\sqrt{3}}{2}$. We know the formula for the volume is one third times the height times the area of the base. The height is the distance from A to the plane, i.e. $\frac{1}{\sqrt{3}}$. So, our answer is $\frac{1}{3} \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{1}{6}$. However, we asked for the volume of the big piece. Since this is a unit cube, the answer is $\frac{5}{6}$.

10. Using the letters that label the cube on the first page, find a non-trivial way to pick up the cube and set it down looking the same (i.e. a transformation), which, when iterated three (3) times puts the cube back where it was. For example, A to C to D to B to A and E to G to H to F to E gives an example of such a transformation that takes four (4) iterations to get back where you started. Note that it is enough to say where one edge goes because the entire rest of the cube must follow, so, for example, the above example of order 4 is completely described by saying that AC goes to CD. Make your own box for your answer.

There are many solutions, for example, AB goes to AC. The key is to go back to that plane that goes through 3 vertices. The plane intersects at an equilateral triangle. Just rotate that triangle.