SOLUTIONS TO PROBLEMS IN

THE WILSON GEOMETRY TEST

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Here are solutions to the problems in the "infamous" geometry test created by W. Stephen Wilson of Johns Hopkins University. Here is the WWW link for the test itself:

`http://www.math.jhu.edu/~wsw/GEOM/geometry.pdf`

The solutions are done here in terms of vector geometry, sometimes in $\mathbb{R}^3$ and other times in $\mathbb{R}^4$. The advantage of working in the latter is that the solid regular tetrahedron can be defined very simply and symmetrically in 4–space as the set of all (x, y, z, w) such that $x, y, z, w \geq 0$ and $x + y + z + w = 1/\sqrt{2}$. Of course, the solid cube can be realized inside $\mathbb{R}^3$ as the set of all (x, y, z) such that $0 \leq x, y, z \leq 1$. We shall take the vertices of the tetrahedron $\mathbf{T_4}$ to be

$$A = \frac{1}{\sqrt{2}}(1,0,0,0)\ , \quad B = \frac{1}{\sqrt{2}}(0,1,0,0)$$

$$C = \frac{1}{\sqrt{2}}(0,0,1,0)\ , \quad D = \frac{1}{\sqrt{2}}(0,0,0,1)$$

and the vertices of the cube $\mathbf{C_6}$ will be given as follows:

$$A = (0,0,1)\ , \quad B = (1,0,1), \quad C = (0,1,1)\ , \quad D = (1,1,1)$$

$$E = (0,0,0)\ , \quad F = (1,0,0), \quad G = (0,1,0)\ , \quad H = (1,1,0)$$

Synthetic derivations of the answers to the questions in the test, accompanied by drawings which illustrate both the synthetic and vector proofs, are available online at the following site:

`http://www.math.jhu.edu/~wsw/GEOM/answers.pdf`

The file `wswGeometrytest.pdf` in this directory contains clickable links for the sites displayed above.

The vector - geometric solutions

1. Let $U = \frac{1}{2}(A+B)$, $V = \frac{1}{2}(B+C)$, $P = \frac{1}{2}(C+D)$, and $Q = \frac{1}{2}(D+A)$. No three of these points are collinear; one way to show this is to display Q as an affine combination of U, V, P such that all of the coefficients are nonzero. But direct computation shows that $Q = U - V + P$. The latter in turn shows that the four points are coplanar; furthermore, we also have $V - U = Q - P$ and $Q - U = P - V$, so that $UV||PQ$ and $QU||PV$. It follows that U, V, Y, Q (in that order) form the vertices of a parallelogram. Furthermore, since $|A|^2 = |B|^2 = |C|^2 = |D|^2 = \frac{1}{2}$ and the four vectors are pairwise orthogonal, it follows that

$$\langle\, (U - V), (V - P)\,\rangle \;=\; \left\langle\, \tfrac{1}{2}(C - A),\, \tfrac{1}{2}(B - D)\,\right\rangle \;=$$

$$\frac{1}{4}\,(\,\cdot\langle C, B\rangle \;-\; \langle A, B\rangle \;-\; \langle C, D\rangle \;+\; \langle A, D\rangle\,)$$

and since the vectors A, B, C, D are pairwise orthogonal it follows that the right hand side is equal to zero, so that $UV \perp VP$; since we have a parallelogram, this implies the parallelogram must be a rectangle. To show it is a square, note that

$$|V-U|^2 \;=\; |\tfrac{1}{2}(A-C)|^2 \;=\; \frac{1}{4}\left(|A|^2 - 2(A\cdot C) + |C|^2\right) \;=\; \frac{1}{4}\left(\frac{1}{2} - 0 + \frac{1}{2}\right) \;=\; \frac{1}{4}$$

$$|V-P|^2 \;=\; |\,\tfrac{1}{2}(B-D)\,|^2 \;=\; \frac{1}{4}\left(|B|^2 - 2(B\cdot D) + |D|^2\right) \;=\; \frac{1}{4}$$

so that U, V, P, Q (in that order) form the vertices of a square whose sides have length $\frac{1}{2}$. The area of the solid region bounded by this square is then equal to $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$.

We shall now explain why the plane of $VPQU$ cuts the solid tetrahedron into two isometric pieces. First of all, by definition the solid tetrahedron lies on the hyperplane H_1 with equation $x+y+z+w = (1/\sqrt{2})$. Also, the plane containing the points U, V, P, Q is the intersection of H_1 with the hyperplane H_2 defined by $x-y+z-w = 0$. Let T be the invertible linear transformation of $\mathbb{R}^4$ given by

$$T(x,y,z,w) \;=\; (y,z,w,x)\;.$$

Then it follows that T maps H_2 onto itself, and in fact T also interchanges the two half-spaces K_+ and K_- defined by the inequalities $x-y+z-w > 0$ and $x-y+z-w < 0$ respectively. Furthermore, it also follows that T maps the original tetrahedron $\mathbf{T_4}$ with vertices A, B, C, D into itself because it maps the set of vertices $\{A, B, C, D\}$ into itself. Therefore the plane H_2 splits the $\mathbf{T_4}$ into two congruent pieces, one of which is $\mathbf{T_4} \cap (K_+ \cup H_2)$ and the other of which is $\mathbf{T_4} \cap (K_- \cup H_2)$.■

2. The given plane contains the point

$$M \;=\; \frac{1}{4}(A+B+C+D) \;=\; \frac{1}{\sqrt{2}}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

and the same is true if we perform the construction of Exercise 1 using the midpoints of but switch the roles of B and C. If we do this, then we get another square which sits in another plane; namely the intersection of the hyperplane with equation $x+y+z+w = 1/\sqrt{2}$ with the hyperplane whose equation is $x-z+y-w = 0$. One can check directly that the three hyperplanes intersect in a line, and the intersection of this line with $\mathbf{T_4}$ will have to be a closed line segment. To complete the solution to the exercise, we must find the endpoints and compute the distance between them.

By construction, V is the midpoint of the closed segment $[BC]$ and Q is the midpoint of the closed segment $[AD]$. If we switch the roles of B and C but leave A and D untouched, then we obtain midpoints which we might call U', V', P' and Q' for the segments $[AC]$, $[CB]$, $[BD]$ and $[DA]$ respectively. Therefore $V = V'$ and $Q = Q'$, and hence the points V and Q lie on both the plane Π_1 containing U, V, P, Q and the plane Π_2 containing U', V', P', Q'.

We claim that the planes Π_1 and Π_2 are distinct. If they were not, then the set of points satisfying the equations for Π_1

$$x+y+z+w \;=\; \frac{1}{\sqrt{2}}\;, \qquad x-y+z-w \;=\; 0$$

will also satisfy the second equation for Π_2; namely, $x-z+y-w = 0$. Therefore all we have to do is find a point which satisfies the first two equations but not the third one; such a choice is given by

$$U \;=\; \frac{1}{2\sqrt{2}}(1,1,0,0)\;.$$

The segment joining V and Q must then be the intersection of the two solid squares in planes Π_1 and Π_2. The length of this segment can be computed for the formulas for V and Q in the first exercise, and it is equal to $1/\sqrt{2}$.■

3. We shall first verify that the 24 possible reorderings of $\{A, B, C, D\}$ yield exactly three distinct planes (the problem does not ask for this, but implicitly assumes that there are exactly three planes). We have already noticed that if we switch B and C but leave A and B alone, the plane we obtain is the intersection of the hyperplane containing $\mathbf{T_4}$ with the hyperplane defined by the equation $x - z + y - w = 0$. In fact, no matter how we reorder the four vertices, the plane we obtain is the intersection of the tetrahedron's hyperplane with one having an equation of the form $x + uy + vz + tw = 0$, where each of u, v, t is ± 1 and exactly one is positive. Since there are only three possible choices for (u, v, w), it follows that there are at most three planes. Some straightforward calculations as before show that these three possibilities are indeed distinct.

We have already noted that the center M of the tetrahedron lies on each of the given planes. To see if there are any other points on all three planes, observe that the common points are given by a system of four equations in four unknowns:

$$x + y + z + w = \frac{1}{\sqrt{2}}\,, \qquad x - y + z - w = 0$$

$$x + y - z + w = 0\,, \qquad x + y + z - w = 0$$

Standard methods from linear algebra show that this system has exactly one solution.■

4. This is probably the easiest of all the exercises to visualize. We simply take the rectangle whose vertices are A, E, H, D. By construction we know that $d(A, E) = 1$, and we also have

$$d(E, H) = |(0, 0, 0) - (1, 1, 0)| \ \sqrt{1 + 1} = \sqrt{2}$$

and therefore the area of the solid square is given by $d(A, E) \cdot d(E, H) = \sqrt{2}$.■

5. Let U be the midpoint of $[BF]$, and let V be the midpoint of $[CG]$, so that $U = (1, 0, \frac{1}{2})$ and $V = (1, 1, \frac{1}{2})$. Then straighforward calculation shows that $d(A, U) = d(U, H) = d(H, V) = d(V, A) = \frac{1}{2}\sqrt{5}$.

In order to verify that we have a rhombus, we need to show that AU is parallel to HV. But if we compute $A - U$ and $H - V$, we find that

$$A - U = \left(-1, 0, \frac{1}{2}\right)\,, \qquad H - V = \left(1, 0, -\frac{1}{2}\right)$$

which means that the lines AU and HV are indeed parallel.■

6. Suppose we are given a rhombus such that the sides all have length a and one of the vertex angles has measure θ. Then the altitude h is given by $a \sin \theta$; it does not matter which vertex angle we choose, for the measures of the other three vertex angles are either θ or $\pi - \theta$, and even in the second case the value of the sine function is the same. For our purposes it is better to write this in the following form:

$$h = a\sqrt{1 - \cos^2 \theta}$$

We know that $a = \frac{1}{2}\sqrt{5}$. Furthermore, the cosine of θ is given by the familiar dot product formula:

$$\cos \theta = \frac{(A - U) \cdot (H - U)}{|A - U| \cdot |H - U|} = \frac{(-1, 0, 1/2) \cdot (0, 1, -1/2)}{5/4} \ \frac{-1/4}{5/4} = -\frac{1}{5}$$

It follows that

$$\sin^2\theta \quad = \quad 1 - \frac{1}{25} \quad = \quad \frac{24}{25}$$

and therefore take square roots and substitute into the formula for h we have

$$h \quad = \quad a\sin\theta \quad = \quad \frac{\sqrt{5}}{2}\cdot\frac{24}{5} \quad = \quad \frac{\sqrt{5}}{2}\cdot\frac{2\sqrt{6}}{5} \quad = \quad \frac{\sqrt{6}}{\sqrt{5}}\ .$$

7. The vertices of the hexagon are midpoints for six of the edges:

P is the midpoint of the closed segment $[AB]$. Its coordinates are $(\frac{1}{2}, 0, 1)$.

Q is the midpoint of the closed segment $[BD]$. Its coordinates are $(1, \frac{1}{2}, 1)$.

S is the midpoint of the closed segment $[FH]$. Its coordinates are $(1, 1, \frac{1}{2})$.

T is the midpoint of the closed segment $[HG]$. Its coordinates are $(\frac{1}{2}, 1, 0)$.

U is the midpoint of the closed segment $[GE]$. Its coordinates are $(0, \frac{1}{2}, 0)$.

V is the midpoint of the closed segment $[EA]$. Its coordinates are $(0, 0, \frac{1}{2})$.

It follows immediately that

$$d(P,Q) \quad = \quad d(Q,S) \quad = \quad d(S,T) \quad = \quad d(T,U) \quad = \quad d(U,V) \quad = \quad d(V,P) \quad = \quad \frac{1}{\sqrt{2}}\ .$$

If we know that P, Q, S, T, U, V (in that order) form the vertices of a regular hexagon, this answers the question in the exercise.

To show that the six points do form the vertices of a regular hexagon, it is necessary to show that the six points are coplanar and that there is some point M on that plane such that

$$d(M,P) \quad = \quad d(M,Q) \quad = \quad d(M,S) \quad = \quad d(M,T) \quad = \quad d(M,U) \quad = \quad d(M,V) \quad = \quad \frac{1}{\sqrt{2}}\ .$$

There is a systematic criterion for deciding whether a set of points $P_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ is coplanar — namely, determining whether the dimension of the span of the vectors $(x_i, y_i, z_i, 1) \in \mathbb{R}^4$ has dimension ≤ 3 — and if this is done by row operations then one can read off the equation of a plane containing the given points, provided such a plane exists. It turns out that the six points all lie on the plane Π defined by the equation $2y + 2z - 2x = 1$. The point M is merely the center of the cube, with coordinates $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$; one can check directly that this point also lies on the plane Π. Furthermore, it turns out that the six difference vectors

$$P - M\ , \quad Q - M\ , \quad S - M\ , \quad T - M\ , \quad U - M\ , \quad V - M$$

each have exactly one zero coordinate and two coordinates with absolute value $\frac{1}{2}$. This means that the distances from M to each of P, Q, S, T, U, V are all equal to $1/\sqrt{2}$, so that we have the following six equilateral triangles:

$$\Delta MPQ\ , \quad \Delta MQS\ , \quad \Delta MST\ , \quad \Delta MTU\ , \quad \Delta MUV\ , \quad \Delta MVP$$

Before we explain why the points P, Q, S, T, U, V are the vertices of a regular hexagon, we must understand more precisely what that means. A regular polygon has a central point, and the preceding discussion suggests that M should be that point. We should then have that M is the midpoint of the closed segments $[PT]$, $[QU]$ and $[SV]$, and in fact these can be checked by direct computation. Furthermore, Q and V should lie on opposite sides of the line containing P, M and T; in fact, the midpoint N of $[QV]$ is given by

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\right) \quad = \quad \frac{3}{4} T \;+\; \frac{1}{4} P$$

so that N lies on the line PT. This means that Q and V must lie on opposite sides of PT in the plane Π_1.

Note that the preceding discussion remains valid if we replace Q, P and V by S, T and U respectively.

Finally, we need to check that Q and S lie on the same side of PT, and similarly U and V lie on the same side of PT; it will follow that the side containing the first two points is opposite the side containing the last two. One way of proving the given statements is to show that the lines SQ and UV are both parallel to PT. But direct calculuation shows that

$$S \,-\, Q \quad = \quad \frac{1}{2}(T \,-\, P) \quad = \quad U \,-\, V$$

and hence the parallelism relations hold. If we combine this with the previous observations, we can conclude that P, Q, S, T, U, V are the vertices of a regular hexagon.■

8. Take the plane which contains the three standard unit vectors, which in our notation are $F = \mathbf{i}$, $G = \mathbf{j}$ and $A = \mathbf{k}$. Then the plane is given by the equation $x + y + z = 1$. The origin $\mathbf{0} = E$ lies on one side because $0 + 0 + 0 < 1$, while the points $B = (1, 0, 1)$, $C = (0, 1, 1)$, $D = (1, 1, 1)$ and $H = (1, 1, 0)$ are on the other side because the sums of their respective coordinates are all positive.

In order to finish the problem, we need to find the distance from $\mathbf{0}$ to this plane. This distance is given by the length of the vector P such that P is perpendicular to the plane and lies on that plane. Now the normal direction to P is given by $(1, 1, 1)$, so the point P has the form (t, t, t) such that $t + t + t = 1$, or $t = \frac{1}{3}$. Therefore we need to compute the distance from the origin to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which is equal to

$$\sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9}} \quad = \quad \sqrt{\frac{1}{3}}\,.■$$

9. The sum of the volumes of the large and small pieces is equal to the volume of the entire cube, which is 1. Since the small piece is a pyramid such that one edge is an altitude, its volume is easy to compute. The volume of the solid plane region bounded by the isosceles right triangle ΔEFG is $\frac{1}{2}$, and therefore we can use the formula

$$\text{(volume)} \quad = \quad \frac{1}{3}\,\text{(base)} \cdot \text{(height)}$$

to see that the volume of the small piece is $\frac{1}{3} \cdot \frac{1}{2} \cdot 1 \;=\; \frac{1}{6}$. Therefore the volume of the large piece is $1 \;-\; \frac{1}{6} \;=\; \frac{5}{6}$.■

10. The idea is to take the invertible linear transformation T on $\mathbb{R}^3$ which permutes $\{\,\mathbf{i},\, \mathbf{j},\, \mathbf{k}\,\}$ cyclically, so that $T(\mathbf{i}) = \mathbf{j}$, $T(\mathbf{j}) = \mathbf{k}$, and $T(\mathbf{k}) = \mathbf{i}$. This map will sends the solid cube into itself, and on vertices it sends E and D to themselves, it permutes $\{F, G, A\}$ cyclically, and it also permutes $\{B, H, C\}$ cyclically.■