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Mathematics 205B, Winter 2021, Examination 1

Answer Key

1. [*25 points*] Suppose that X is the union of the arcwise connected open sets $U \cup V$ such that $U \cap V$ is also arcwise connected. Assume further that the homomorphismss $\pi_1(U \cap V) \to$ (*both*) $\pi_1(U)$, $\pi_1(V)$ induced by inclusion are isomorphisms. Prove that the maps from both $\pi_1(U)$ and $\pi_1(V)$ to $\pi_1(X)$ induced by inclusion are also isomorphisms.

SOLUTION

Let G_0, G_1 and G_2 be $\pi_1(U \cap V)$, $\pi_1(U)$ and $\pi_1(V)$ respectively, and let $f : G_0 \to G_1$ and $g : G_0 \to G_2$ be the induced maps of fundamental groups; our assumptions imply that these maps are group isomorphisms.

By Van Kampen's Theorem we know that $\pi_1(X)$ is isomorphic to the pushot in the following diagram, in which f and g are isomorphisms:

$$\begin{array}{ccc} G_0 & \xrightarrow{\ f\ } & G_1 \\ \Big\downarrow g & & \Big\downarrow \\ G_2 & \longrightarrow & P \end{array}$$

It suffices to show that the Universal Mapping Property for pushouts holds if we substitute G_0 for P and take the maps $G_1 \to P$ and $G_2 \to P$ to be f^{-1} and g^{-1} respectively.

To prove this, we need to show that if we are given group homomorphisms $p : G_1 \to H$ and $q : G_2 \to H$ satisfying $p \circ f = q \circ g$, then there is a unique homomorphism $k : G_0 \to H$ such that $k \circ f^{-1} = p$ and $k \circ g^{-1} = q$. If we let k be the composite $p \circ f = q \circ g$ then we have the desired identities $k \circ f^{-1} = p$ and $k \circ g^{-1} = q$ by the assumptions. This proves existence; to prove uniqueness, suppose that $j : G_0 \to H$ satisfies $j \circ f^{-1} = p$ and $j \circ g^{-1} = q$. If we right multiply both sides of the first identity by f, we find that $j = p \circ = k$. Therefore our diagram satisfies the conditions for a pushout.■

2. [*25 points*] (a) If the following statement is true, give a proof; if it is false, give a counterexample: *Suppose that the graph T is a tree and F is an edge in T. Then the subgraph formed by all edges except F is a tree.*

(b) Give an example of a connected graph with exactly one maximal tree, and give an example of a connected graph with at least two maximal trees.

SOLUTION

(a) Here is a simple counterexample: Take X to be the union of the three edges $[0,1]$, $[1,2]$ and $[2,3]$, so that $X = [0,3]$ is a tree. If we remove $F = [1,2]$ then the remaining subgraph is the disconnected set $[0,1] \cup [2,3]$. Since a tree is connected, the remaining subgraph is not a tree.■

(b) Every tree is (tautologically) a maximal tree; the simplest example is the one edge graph $[0,1]$. If we take the boundary of a 2-simplex and remove any one of the edges, the remaining subgraph is a maximal tree. Since there are three edges in the original graph, there are three maximal trees given by this process. More generally, if we take the graph obtained from a regular n-gon and remove one edge, the remaining subgraph is a tree, and hence we have n maximal trees in this case.■

3. [*25 points*] Let W be a simply connected space, let X be a connected graph, and let $Y \to X$ be a finite covering. Prove that every continuous map $W \to X$ lifts to a continuous mapping from W to Y. You may assume all spaces are also Hausdorff and locally arcwise connected.

SOLUTION

Choose an arc component $Y_0 \subset Y$; then the restriction $q : Y_0 \to X$ is also a finite covering. Choose $w_0 \in W$ and $y_0 \in Y_0$ such that $f(w_0) = x_0$ lifts to y_0 in Y_0 (one can always find such a point by the Path Lifting Property). By the Lifting Criterion there is a continuous lifting $F : (W, w_0) \to (Y_0, y_0)$ if and only if the image of the homomorhism $f_* : \pi_1(W, w_0) \to \pi_1(X, x_0)$ is contained in the image of $q_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$. Since the fundamental group of W is assumed to be trivial, the image of $f_*$is the trivial group, which is contained in the image of q_* regardless of what the latter image might be, and therefore we know that f lifts to a map $f_0 : W \to Y_0$. The lifting to Y is simply the composite of f_0 with the inclusion $Y_0 \subset Y$.∎

4. [*25 points*] Suppose that we are given a solid regular n-gon X in the plane, where $n \geq 4$, and let $v_1, \cdots, v_n$ are its vertices so that the boundary is the closed path $x_1x_2 + x_2x_3 + ... + x_nx_1$. Assume the vertices are ordered as indicated; denote this ordering by ω. Let $\mathbf{K}$ be the triangulation of X by the 2-simplices $x_ix_{i+1}x_n$ where $i = 1, ..., n-2$. A drawing for $n = 7$ is given on the next page. Find a simplicial 2-chain in $C_2(X, \mathbf{K}, \omega)$ whose boundary is a sum of terms

$$\varepsilon_1 x_1x_2 \ + \ \varepsilon_2 x_2x_3 \ + \ \cdots \ \varepsilon_{n-1}x_{n-1}x_n \ + \ \varepsilon_n x_1x_n$$

where $\varepsilon_i = \pm 1$. Prove that the boundary of your 2-chain has the required property.

SOLUTION

By definition we know that

$$d(x_ix_{i+1}x_n) \quad = \quad x_ix_{i+1} \ + \ x_{i+1}x_n \ - \ x_ix_n$$

and if we take the sum of these for $i = 1, ..., n-2$ we find that

$$d\left(\sum_{i=1}^{n-2} x_ix_{i+1}x_n\right) \quad = \quad \left(\sum_{k=i}^{n-1} x_ix_{i+1}\right) \ - \ x_1x_n$$

which is the sort of answer one expects (the contributions from terms of the form x_ix_n cancel each other out unless $i = 1$ or $n-1$.∎