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Mathematics 205B, Winter 2021, Examination 2

Answer Key

1. [*25 points*] Let $P = \Delta_2 \times I$ denote the standard solid 3-dimensional triangular prism, with ordered vertices $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ (bottom) and $\mathbf{y}_0$, $\mathbf{y}_1$, $\mathbf{y}_2$ (top); the boundary $\partial(\Delta_2 \times I) = (\Delta_2 \times \{0,1\}) \cup (\partial\Delta_2 \times I)$ is then a subcomplex of the given simplicial decomposition. Find a chain $A \in C_2(\partial(\Delta_2 \times I), \mathrm{decomp}, \omega)$ such that A is a linear combination of every 2-simplex in ∂P, the coefficient of each free generator is ± 1, and $d_2(A) = 0$.

SOLUTION

The chain is suggested by the drawing on the next page. We want to choose signs so that the boundaries of the individual simplices cancel each other. Since the 2-chain A has the required properties if and only if $-A$ does, we may assume that the coefficient of one free generator for $C_2(\partial(\Delta_2 \times I), \mathrm{decomp}, \omega)$ is $+1$. We shall stipulate that A be chosen so that the coefficient of the free generator $\mathbf{y}_0\,\mathbf{y}_1\,\mathbf{y}_2 \in C_2$ is $+1$.

With this condition, direct computation shows that A must be equal to

$$\mathbf{y}_0\,\mathbf{y}_1\,\mathbf{y}_2 \;+\; \mathbf{x}_0\,\mathbf{y}_0\,\mathbf{y}_2 \;-\; \mathbf{x}_1\,\mathbf{y}_1\,\mathbf{y}_2 \;-\; \mathbf{x}_0\,\mathbf{y}_0\,\mathbf{y}_1 \;-$$

$$\mathbf{x}_0\,\mathbf{x}_2\,\mathbf{y}_2 \;+\; \mathbf{x}_1\,\mathbf{x}_2\,\mathbf{y}_2 \;+\; \mathbf{x}_0\,\mathbf{x}_1\,\mathbf{y}_1 \;-\; \mathbf{x}_0\,\mathbf{x}_1\,\mathbf{x}_2$$

because its boundary is equal to

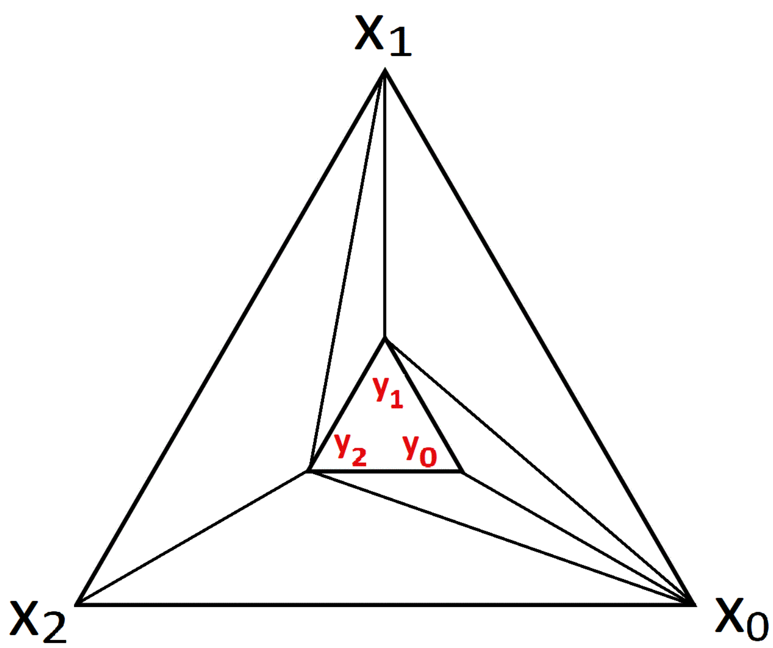
$$\mathbf{y}_1\,\mathbf{y}_2 \;-\; \mathbf{y}_0\,\mathbf{y}_2 \;+\; \mathbf{y}_0\,\mathbf{y}_1 \;+\; \mathbf{y}_0\,\mathbf{y}_2 \;-\; \mathbf{x}_0\,\mathbf{y}_2 \;+\; \mathbf{x}_0\,\mathbf{y}_0 \;-\; \mathbf{y}_1\,\mathbf{y}_2 \;+\; \mathbf{x}_1\,\mathbf{y}_2 \;-\; \mathbf{x}_1\,\mathbf{y}_1 \;-$$

$$\mathbf{x}_0\,\mathbf{y}_0 \;+\; \mathbf{x}_0\,\mathbf{y}_1 \;-\; \mathbf{y}_0\,\mathbf{y}_1 \;-\; \mathbf{x}_0\,\mathbf{x}_2 \;+\; \mathbf{x}_0\,\mathbf{y}_2 \;-\; \mathbf{x}_2\,\mathbf{y}_2 \;+\; \mathbf{x}_1\,\mathbf{x}_2 \;-\; \mathbf{x}_1\,\mathbf{y}_2 \;+\; \mathbf{x}_2\,\mathbf{y}_2 \;+$$

$$\mathbf{x}_0\,\mathbf{x}_1 \;-\; \mathbf{x}_0\,\mathbf{y}_1 \;+\; \mathbf{x}_1\,\mathbf{y}_1 \;-\; \mathbf{x}_1\,\mathbf{x}_2 \;+\; \mathbf{x}_0\,\mathbf{x}_2 \;-\; \mathbf{x}_0\,\mathbf{x}_1$$

and the 24 terms in this expression cancel in pairs.∎

Drawing for Problem 1



2. [*25 points*] (a) Show that $(\mathbb{R}^2 \times \{0\}) \cup (\{0,0) \times \mathbb{R}) \subset \mathbb{R}^3$ is not homeomorphic to $\mathbb{R}^2$. [*Hint:* Look at local propeties at the origin. Any valid method is acceptable.]

(b) Show that the standard inclusion $S^1 \times S^1 \subset S^3$ is not a retract.

SOLUTION

(a) The simplest way to prove this is to notice that the complement of the origin in $X = (\mathbb{R}^2 \times \{0\}) \cup (\{0,0)\} \times \mathbb{R})$ is disconnected. This subspace is the union of the disjoint subsets

$$(\mathbb{R}^2 \times \{0\}) - \{(0,0,0)\} \quad \text{and} \quad \{(0,0)\} \times (\mathbb{R} - \{0\}) \ .$$

We claim that both of these subsets are open in X; this is true because both $(\mathbb{R}^2 \times \{0\})$ and $(\{(0,0)\} \times \mathbb{R})$ are closed subsets of $\mathbb{R}^3$ (hence also in X) and

(i) $(\mathbb{R}^2 \times \{0\}) - \{(0,0,0)\}$ is the relative complement of $\{(0,0)\} \times \mathbb{R}$ in X,

(ii) $\{(0,0)\} \times (\mathbb{R} - \{0\})$ is the relative complement of $\mathbb{R}^2 \times \{0\}$ in X.

On the other hand, the complement of a point in $\mathbb{R}^2$ is always homeomorphic to the connected set $S^1 \times \mathbb{R}$. Since this does not hold for X, it follows that X and $\mathbb{R}^2$ are not homeomorphic.■

(b) Pick a base point $p \in S^1 \times S^1$. If the inclusion is a retract then the associated map of fundamental groups $\pi_1(S^1 \times S^1, p) \to \pi_1(S^3, p)$ will be 1–1. Since the fundamental groups of the spaces are isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and the trivial group, this is not the case. Therefore there cannot be a retraction $\rho : S^3 \to S^1 \times S^1$ such that $\rho | S^1 \times S^1$ is the identity.■

3. [*25 points*] Suppose that A is a nonempty subspace of the topological space X, and let $i : A \to X$ denote the inclusion. Prove that all the maps in homology $i_* : H_q(A) \to H_q(X)$ are isomorphisms if and only if all of the relative homology groups $H_q(X, A)$ are trivial. [*Hint:* What does it mean to have zero mappings in an exact sequence?]

SOLUTION

We shall use the long exact homology sequence containing the groups $H_q(X, A)$, where q denotes an arbitrary integer:

$$\cdots \quad H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \xrightarrow{i_*} H_{q-1}(X) \quad \cdots$$

If $H_*(X, A) = 0$ in all dimensions, then all of the mappings j_* and ∂ must be trivial. Since the kernel of j_* equals the image of i_* by exactness, it follows that $H_q(X) = \operatorname{Ker} j_* = \operatorname{Image} i_*$ and hence the mappings i_* are onto. Likewise, since the kernel of i_* equals the image of *partial* by exactness, it follows that $0 = \operatorname{Image} \partial = \operatorname{Ker} i_*$ and hence the mappings i_* are also 1–1. Combining these, we conclude that the mappings i_* are isomorphisms.■

Conversely, assume that all the mappings i_* are isomorphisms. By exactness we have $H_q(X) = \operatorname{Image} i_* = \operatorname{Ker} j_*$ and $0 = \operatorname{Ker} i_* = \operatorname{Image} \partial$. Therefore the mappings j_* and ∂ are all trivial. We must now show that these imply $H_*(X, A) = 0$ in all dimensions.

Let $u \in H_q(X, A)$ for some q. Then $\partial = 0$ implies that $\partial(u) = 0$ and hence $u = j_*(v)$ for some v, and since $j_* = 0$ it follows that $u = 0$. Therefore $H_q(X, A) = 0$ for all q.■

4. [*25 points*] (*a*) Suppose we are given a subset $A \subset S^3$ which is a union of three compact subsets $B_1 \cup C \cup B_2$ where B_1 and B_2 are disjoint subsets which are homeomorphic to S^2 and C is homeomorhic to a closed interval such that each intersection $C \cap B_i$ is an endpoint of C. Prove that the complement $S^3 - A$ has three connected components. [*Hint:* What is the reduced homology of $S^3 - (B_i \cup C)$ for $i = 1, 2$?]

EXTRA CREDIT. [*10 points*] For each component Ω as above, state a conjecture about which points of A should be limit points of Ω.

SOLUTION

We shall need a few Mayer-Vietoris exact sequences in singular homology:

$$\cdots \to \widetilde{H_{q+1}}(U \cup V) \to \widetilde{H_q}(U \cap V) \to \widetilde{H_q}(U) \oplus \widetilde{H_q}(V) \to \widetilde{H_q}(U \cup V) \to \cdots$$

Following the hint, we shall first apply this to $S^3 - (B_i \cup C) = (S^3 - B_i) \cap (S^3 - C)$ where $i = 1, 2$. Specificially, let $U_i = S^3 - B_i$ and $V = S^3 - C$, and let $B_i \cap C = \{p_i\}$, so that $U_i \cup V = S^3 - \{p_i\}$. Then the reduced homology groups of V and $U_i \cup V$ are trivial, the first by a theorem in the notes and the second because $U \cup V = S^3 - \{p\} \cong \mathbb{R}^3$. Therefore the exact Mayer-Vietoris sequence implies that the inclusion map induces homology isormorphisms

$$\widetilde{H_*}\left(U_i \cap V = S^3 - (B_i \cup C)\right) \longrightarrow \widetilde{H_*}(U_i) \ .$$

Next, consider the Mayer-Vietoris sequence for U_1 and U_2; in this case $U_1 \cap U_2 = S^3 - A$ and $U_1 \cup U_2 = V$. In this case the exact Mayer-Vietoris sequence and the triviality of $\widetilde{H_*}\left(U_1 \cup U_2 = (S^3 - C)\right)$ imply that the inclusion maps induce isomorphisms

$$\widetilde{H_q}(U_1 \cap U_2) \to \widetilde{H_q}(U_1) \oplus \widetilde{H_q}(U_2) \ .$$

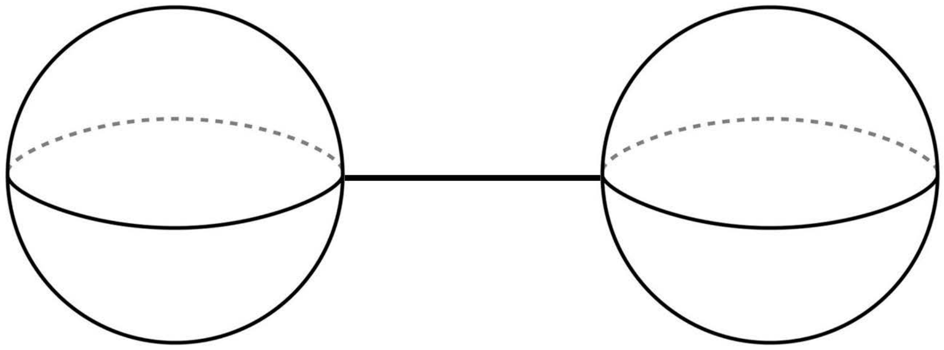
The right hand side is trivial if $q \neq 0$ and isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ if $= 0$. Therefore

$$H_0\left(S^3 - A = (U_1 \cup U_2)\right) \quad \cong \quad \mathbb{Z}^3$$

which means that $S^3 - A$ has exactly three (connected or arcwise) components.∎

A drawing for this problem and a discussion of the extra credit question appear on the next page.

Drawing for Problem 4



SOLUTION TO EXTRA CREDIT QUESTION

The drawing suggests that the three components of the complement $S^3 - A$ have the following frontiers:

(i) The frontier of one component is B_1, and the frontier of a second component is B_2.

(ii) The frontier of the third component is equal to A. ■

5. [*25 points*] Let U and V be (arcwise) connected open subsets of $\mathbb{R}^n$ such that $U \cap V$ is a nonempty convex set and $\pi_1(U \cup V)$ is finite. Prove that at least one of U and V must be simply connected. [*Hint:* What is the contrapositive?]

SOLUTION

The contrapositive statement is that if both U and V are not simply connected then $\pi_1(U \cup V)$ is infinite. We know that $U \cap V$ is simply connected, so by Van Kampen's Theorem we also know that $pi_1(U \cup V)$ is a free product of $\pi_1(U)$ and $\pi_1(V)$. Therefore the proof of the contrapositive reduces to showing that if G and H are nontrivial groups, then the free product $G * H$ is infinite. Let $i_G : G \to G * H$ and $i_H : H \to G * H$ be the images of the two groups in their free product.

To prove the latter, let $1 \neq g \in G$ and $1 \neq h \in H$. Then we know that the element $i_G(g) \cdot i_H(h) \in G * H$ has infinite order because every nontrivial element in the free product is a monomial in which the images of nontrivial elements in the images of i_G and i_H always appear in an alternating pattern.■