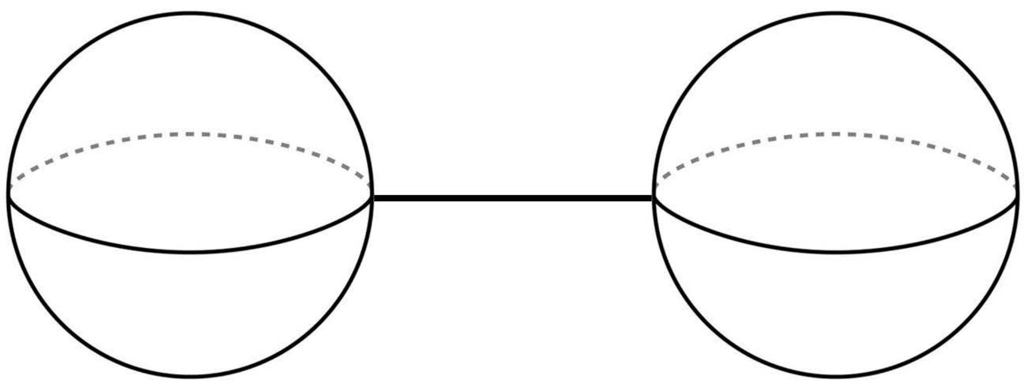
Justifying the answer to the Extra Credit for Problem 4 in Examination 2

Recall the setting: We are given a subset $A \subset S^3$ which is a union of three compact subsets $B_1 \cup C \cup B_2$ where B_1 and B_2 are disjoint subsets which are homeomorphic to S^2 and C is homeomorhic to a closed interval such that each intersection $C \cap B_i$ is an endpoint of C. The solution to the main problem shows that the complement $S^3 - A$ has three connected components.

The extra credit problem is to state a conjecture about which points of A should be limit points of Ω for each component Ω as above, and the correct answer is that the three components of the complement $S^3 - A$ have the following frontiers:

(i) The frontier of one component is B_1, and the frontier of a second component is B_2.

(ii) The frontier of the third component is equal to A.



Proof of the assertion. The first step is to analyze the complement of $B_1 \cup B_2$, and we shall do this using the Jordan-Brouwer Separation Theorem for $S^3 - B_1$ and $S^3 - B_2$. The latter results imply that both $S^3 - B_1$ and $S^3 - B_2$ have two components:

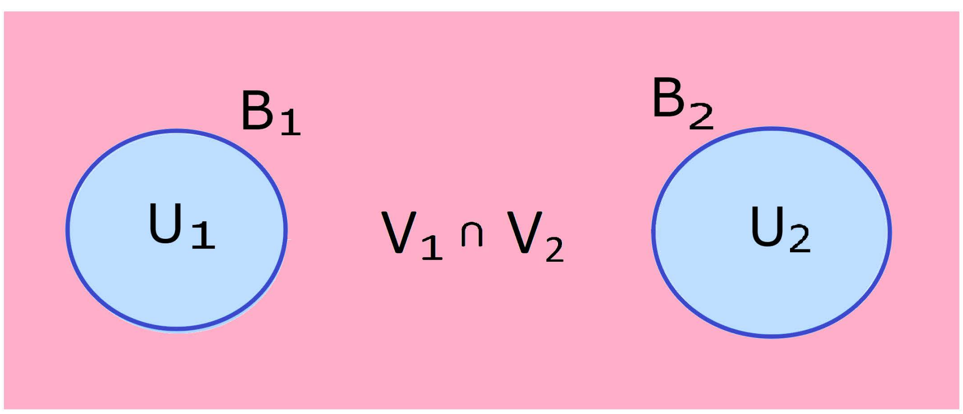
$$S^3 - B_1 \quad = \quad U_1 \ \cup \ V_1 \ , \qquad S^3 - B_2 \quad = \quad U_2 \ \cup \ V_2$$

Since $B_1 \subset S^3 - B_2 = U_2 \cup V_2$ and each of B_1, U_2, V_2 is connected, we know that either $B_1 \subset U_2$ or $B_1 \subset V_2$ (but not both). Similarly, we know that either $B_2 \subset U_1$ or $B_2 \subset V_1$ (but not both). Without loss of generality, we may assume that the components of $S^3 - B_1$ and $S^3 - B_2$ are labeled so that $B_2 \subset V_1$ and $B_1 \subset V_2$.

CLAIM 1: We also have $U_2 \subset V_1$ and $U_1 \subset V_2$.

If we can prove the first inclusion, the second one follows by switching X_1 and X_2 for $X = B, U, V$. Since every point $x \in B_2$ is a limit point of both U_2 and V_2, if we take an open neighborhood N of x which s contained in V_1 then $N \cap U_2 \neq \emptyset$; by local connectedness we may assume that N is (arcwise) connected. By assumption $x \in V_1$, and therefore the set $V_1 \cup B_2 \cup U_2$ is a connected subset of $S^3 - B_1$. But V_1 is a maximal connected subset of the latter and therefore it follows that $U_2 \subset V_1$.■

CLAIM 2: The connected components of $S^3 - (B_1 \cup B_2)$ are given by U_1, U_2 and $V_1 \cap V_2$. Furthermore, the frontier of $V_1 \cap V_2$ is equal to $B_1 \cup B_2$.



We have two representations of S^3 as a union of pairwise disjoint subsets:

$$S^3 \;=\; U_1 \;\cup\; B_1 \;\cup\; V_1 \;, \qquad S^3 \;=\; U_2 \;\cup\; B_2 \;\cup\; V_2$$

If we take the intersections of the expressions on the right hand sides of these identities, we obtain a desription of S^3 as a union of 9 pairwise disjoint subsets in which many of the terms simplify, and this yields the following description of S^3 as a union of pairwise disjoint subsets:

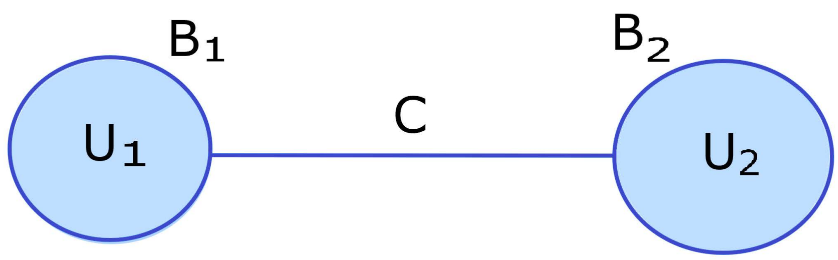
$$S^3 \;=\; U_1 \;\cup\; B_1 \;\cup\; (V_1 \cap V_2) \;\cup\; B_2 \;\cup\; U_2$$

By the first part of the solution to Problem 4, we know that $S^3 - (B_1 \cup B_2)$ has 3 components, and we also know that both U_1 and U_2 are connected open and closed subsets of $S^3 - (B_1 \cup B_2)$ (note that the closure of U_i in S^3 is just $U_i \cup B_i$). It follows that U_1, U_2

and the open and closed subset $V_1 \cap V_2 \subset S^3 - (B_1 \cup B_2)$ must be the components of the latter.

To complete the proof of the claim, we must show that the frontier of $V_1 \cap V_2$ in S^3 is equal to $B_1 \cup B_2$. The displayed decomposition of S^3 as a union of 5 pairwise disjoint subsets implies that the frontier is contained in $B_1 \cup B_2$, and as before it will suffice to show that every point of B_1 is a limit point of $V_1 \cap V_2$ (the other case will follow by switching the roles of 1 and 2). Let $x \in B_1$ and let W be an open neighborhood of x in S^3. Then $W \cap V_2$ is also an open neighborhood of x since $B_1 \subset V_2$, and therefore by the Jordan-Brouwer Separation Theorem there is a point $y \neq x$ in $W \cap V_2 \cap V_1$, so that x is a limit point of $V_1 \cap V_2$ as desired.■

CLAIM 3: If $C \subset A$ is the curve described above then $C - \{p, q\}$ is contained in $V_1 \cap V_2$. Furthermore, if $U_3 = (V_1 \cap V_2) - C$, then U_3 is connected and every point of A is a limit point of U_3.



The set $C - \{p, q\}$ is a connected subset which is contained in $S^3 - (B_1 \cup B_2)$, so it is contained in exactly one of U_1, U_2 and $V_1 \cap V_2$. Furthermore, the compact set C is the closure of $C - \{p, q\}$ because p and q are limit points of the latter. If $C - \{p, q\} \subset U_1$, then the both limit points p and q must belong to the closure of U_1, which by the Jordan-Brouwer Separation Theorem is equal to $U_1 \cup B_1$. This is impossible because one of $\{p, q\}$ is contained in B_1 and the other is contained in B_2. Similarly, we cannot have $C - \{p, q\} \subset U_2$, and hence the only remaining possibility is $C - \{p, q\} \subset V_1 \cap V_2$.

Next, let $B_i^* = B_i \cup C$ where $i = 1, 2$, so that $A = B_1^* \cup B_2^*$ and $C = B_1^* \cap B_2^*$. If we now write $B = B_1 \cup B_2$, then we have the following commutative diagram whose rows are reduced Mayer-Vietoris exact sequences:

$$\begin{array}{ccccccc}
0 = \widetilde{H}_1(S^3 - C) & \rightarrow & \widetilde{H}_0(S^3 - A) & \rightarrow & \bigoplus_{i=1}^2 \widetilde{H}_0(S^3 - B_i^*) & \rightarrow & \widetilde{H}_0(S^3 - C) = 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 = \widetilde{H}_0(S^3) & \rightarrow & \widetilde{H}_0(S^3 - B) & \rightarrow & \bigoplus_{i=1}^2 \widetilde{H}_0(S^3 - B_i) & \rightarrow & \widetilde{H}_0(S^3) = 0
\end{array}$$

In this digram we know that the horizontal maps in the middle are isomorphisms by exactness, and we also know that all the vertical maps are isomorphisms except possibly

$\widetilde{H}_0(S^3 - A) \to \widetilde{H}_0(S^3 - B)$. By the commutativity of the diagram it follows that the latter map must also be an isomorphism. This means that $U_3 = (V_1 \cap V_2) - C$ is also a connected (open) subset of $S^3 - C$ and hence is one of the latter's connected components.

Finally, we need to show that every point of A is a limit point of U_3. The crucial step is to show that the relatively closed subset $C - \{p, q\} \subset V_1 \cap V_2$ is nowhere dense; in other words, its interior is empty.

Assume that some point $z \in C - \{p, q\}$ is in the interior of $V_1 \cap V_2$. Since $C - \{p, q\}$ is homeomorphic to $(0, 1)$, its local homology at z must be $\mathbb{Z}$ in dimension 1 and zero otherwise. On the other hand, since z lies in the interior of $C - \{p, q\}$, its local homology in the latter is equal to its local homology in U_3. Now the local homology in the latter is $\mathbb{Z}$ in dimension 3 and zero otherwise, so we have a contradiction. The source of the contradiction is our assumption that z lies in the interior of $C - \{p, q\}$, and hence it follows that the interior of the latter in U_3 must be empty. Therefore $C - \{p, q\}$ is nowhere dense in $V_1 \cap V_2$ and hence every point of $C - \{p, q\}$ is a limit point of U_3.

To conclude the argument, we must also verify that every point of $B_1 \cup B_2$ is a limit point of U_3. Recall that the second claim implies that every point of $B_1 \cup B_2$ is a limit point of $V_1 \cap V_2$, and by definition we have $U_3 = (V_1 \cap V_2) - C$. Suppose now that $x \in B_1$ and W is an open neighborhood of x in S^3. By the second claim there is a point $y \in W \cap (V_1 \cap V_2)$ (since the intersection is disjoint from B_1 we know that $y \neq x$. If $y \notin C - \{p, q\}$ we are done, but if $y \in C - \{p, q\}$ then by the preceding paragraph we know that there is a point $z \neq y$ such that $z \in W \cap (V_1 \cap V_2) - C = W \cap U_3$; as before, since $x \notin U_3$ it also follows that $z \neq x$. Therefore x is a limit point of U_3, and hence every point of B_1 is a limit point of U_3. If we replace B_1 with B_2 in this argument, we also see that every point of B_2 is a limit point of U_3.

Combining these observations, we see that every point of $A = B_1 \cup C \cup B_2$ is a limit point of U_3.■