EXERCISES FOR MATHEMATICS 205A — PART TWO

FALL 2014

The references denote sections of the textS for the course:

[**M**] J. R. Munkres, *Topology* (Second Edition), Prentice-Hall, Saddle River NJ, 2000. ISBN: 0–13–181629–2.

[**H**] **A. Hatcher.** *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0–521–79540–0.

Solutions to nearly all the exercises below are given in separate files called `solutions`n`.pdf` (in the course directory) for $n \geq 7$. Here is another web site with solutions to exercises in Munkres (including some not given in our files):

`http://www.math.ku.dk/∼moller/e03/3gt/3gt.html`

VII. Topological deformations and approximations

VII.0 : Categories and functors

(Course directory, `categories.pdf`)

Additional exercises

1. Definition. A morphism $f : A \to B$ in a category is a *monomorphism* if for all $g, h : C \to A$ we have that $f \circ h = f \circ g$ implies $h = g$. Dually, a morphism $f : A \to B$ in a category is an *epimorphism* if for all $u, v : B \to D$ we have that $u \circ f = v \circ f$ implies $u = v$.

(a) Prove that a monomorphism in the category **Sets** is $1 - 1$ and an epimorphism in **Sets** is onto. [*Hint:* Prove the contrapositives.]

(b) Prove that in the category of Hausdorff topological spaces (and continuous maps) a morphism $f : A \to B$ is an epimorphism if $f[A]$ is dense in B.

(c) Prove that the composite of two monomorphisms is a monomorphism and the composite of two epimorphisms is an epimorphism.

(d) A morphism $r : X \to Y$ in a category is called a *retract* if there is a morphism $q : Y \to X$ such that $q \circ r = \mathrm{id}_X$. For example, in the category of sets or topological spaces the diagonal map $d_X : X \to X \times X$ is a retract with $q =$ projection onto either factor. Prove that every retract is a monomorphism.

(e) A morphism $p : A \to B$ in a category is called a *retraction* if there is a morphism $s : B \to A$ such that $q \circ r = \mathrm{id}_B$. For example, if r and q are as in (d) then q is a retraction. Prove that every retract is an epimorphism.

2. Let $\mathbf{A}$ be a category, and let $f : A \to B$ be a morphism in $\mathbf{A}$ such that the induced map of morphisms

$$\mathrm{Mor}(f, \cdots) : \ \mathrm{Mor}(B, \cdots) \quad \longrightarrow \quad \mathrm{Mor}(A, \cdots)$$

which sends a morphism $h : B \to C$ to the composite $h \circ f$, is an isomorphism for all objects C in $\mathbf{A}$. Prove that f is an isomorphism. [*Hint:* Choose $C = B$ or A and consider the preimages of the identity elements.] Also prove the (relatively straightforward) converse.

3. An object $\mathcal{O}$ is called an *initial object* in the category $\mathbf{A}$ if for each object A in $\mathbf{A}$ there is a unique morphism $\mathcal{O} \to A$. An object $\mathcal{T}$ is a *terminal object* in $\mathbf{A}$ if for each object A there is a unique morphism $A \to \mathcal{T}$.

(a) Prove that the empty set is initial and every one point set is terminal in **Sets**.

(b) Prove that a zero-dimensional vector space is both initial and terminal in the category $\mathbf{Vec}(\mathbb{F})$ of vector spaces over a field $\mathbb{F}$.

(c) Prove that every two initial objects in a category $\mathbf{A}$ are uniquely isomorphic (there is a unique isomorphism from one to the other), and similarly for terminal objects.

(d) If $\mathbf{A}$ contains an object Z that is both initial and terminal, prove that for each pair of objects A, B in $\mathbf{A}$ there is a unique morphism $A \to B$ that factors as $A \to Z \to B$. Also, if W is any other such object, prove that this composite equals the composite $A \to W \to B$. [*Hint:* Consider the unique maps from W to Z and vice versa.]

Note. An object which is both initial and terminal is called a *null object* (so the null objects in the category of vector spaces are the vector spaces which consist only of the zero element).

4. Prove that a covariant functor takes retracts to retracts and retractions to retractions. State the corresponding result for contravariant functors (the statement is slightly more complicated!).

5. If E is a terminal object in the category $\mathbf{A}$ and $f : E \to X$ is a morphism in $\mathbf{A}$, prove that f is a monomorphism (in fact, something stronger is true—what is it?).

6. Let $\mathbf{A} = (\mathbb{N}^+, \mathrm{Mor}, \varphi)$, where $\mathbb{N}^+$ denotes the positive integers, $\mathrm{Mor}(p, q)$ denotes all $p \times q$ matrices with integer coefficients, and

$$\varphi \ : \ \mathrm{Mor}(p, q) \times \mathrm{Mor}(q, r) \quad \longrightarrow \quad \mathrm{Mor}(p, r)$$

is matrix multiplication. Verify that $\mathbf{A}$ is a category.

7.* If f is a morphism in a category $\mathbf{A}$, a morphism g (in the same category) is called a *quasi-inverse* for f if and only if $f \circ g \circ f = f$. Prove that every morphism that has a quasi-inverse is itself the quasi-inverse of some morphism in the category.

VII.1 : Homotopic mappings

([**M**], §§ 51, 52; [**H**], Ch. 0, § 1.1)

Munkres, § 51, p. 330: 2, 3

Munkres, § 52, p. 334: 1a

Additional exercises

1. Let X be a topological space, and let P be a topological space consisting of exactly one point (it has a unique topology). Explain why the set of homotopy classes $[P, X]$ is in 1–1 correspondence with the set of arc components of X.

2. Let Y be a nonempty topological space with the indiscrete topology (*i.e.*, $\emptyset$ and Y are the only open sets), and let X be an arbitrary nonempty topological space. Prove that $[X, Y]$ consists of a single point. [*Hint:* For all topological spaces W, every map of sets from W to Y is continuous. Using this, show that if $A \subset B$ is a subspace and $g : A \to Y$ is continuous, then g extends to a continuous map from B to Y.]

3. Let Y be a nonempty space with the discrete topology (all subsets are open), and let X be a nonempty connected space. Prove that there is a 1–1 correspondence between $[X, Y]$ and Y.

4. (*i*) Show that if A is a star convex subset of $\mathbb{R}^n$ in the sense of Munkres, Exercise 1, page 334, then the identity map is homotopic to the constant map which sends every point to the "focal point" a_0 (by definition, this is the point such that for each $x \in A$ the closed segment joining x to a_0 lies in A).

(*ii*) If A is a convex subset of $\mathbb{R}^n$, then it follows that A is star convex with respect to every point $a_0 \in A$, but the converse is false. Show that an explicit counterexample is given by considering

$$A \quad = \quad [-2, 2] \times [-1, 1] \ \cup \ [-1, 1] \times [-2, 2] \quad \subset \quad \mathbb{R}^2$$

and showing that A is star convex with respect to $(0, 0)$, but A is not convex. [*Hints:* More generally, it K and L are convex subsets of $\mathbb{R}^n$ and $p \in K \cap L$, then $K \cup L$ is star convex with respect to p. To see that A is not convex, consider the points $(7/4, 3/4)$ and $(3/4, 7/4)$; what is their midpoint? It will probably help to draw a picture of A).

VII.2 : Some examples

Additional exercises

1. Let U be an open subset in $\mathbb{R}^n$ for some n. Prove that U has countably many arc components. [*Hint:* Why is every point in the same arc component as a point with rational coordinates?]

2. Suppose that f_0 and f_1 are homotopic continuous mappings from one space X to another space Y. Prove that for each arc component $A \subset X$ the images $f_0[A]$ and $f_1[A]$ lie in the same arc component of Y.

VII.3 : Homotopy classes of mappings

([**M**], §§ 51, 52, 58; [**H**], Ch. 0)

Munkres, § 58, pp. 366–367: 6

Hatcher, pp. 18–20: 4, 10, 12

Additional exercises

1. If X and Y are topological spaces and $f, g : X \to Y$ are homotopic homeomorphisms, prove that their inverses f^{-1} and g^{-1} are also homotopic. Note that if H is a homotopy from f to g and $t \in [0,1]$, then the map $h_t : X \to Y$ given by $h_t \leftrightarrow H|X \times \{t\}$ is not necessarily a homeomorphism.

2. (i) Suppose that $A \subset X$ and the inclusion mapping $i : A \to X$ is a retract. Prove that for every space Y the induced map of homotopy classes $i_* : [Y,A] \to [Y,X]$ is 1–1. Also, prove that this does not necessarily hold for inclusion maps which are not retracts using the example $Y = A = \{0,1\} \subset [0,1] = X$. [*Hints:* For the first part, if $r : X \to A$ is such that $r \circ i = \mathrm{id}_A$, what can we say about $r_* \circ i_*$? For the second part, first explain why there is a 1–1 correspondence between set-theoretic self-maps of A and $[A,A]$, then explain why $[A,X]$ consists of a single point.]

(ii) In the same setting as (i), prove that for every space Z the induced map of homotopy classes $i^* : [Z,X] \to [Z,A]$ is onto, and give an example to show this is not necessarily true for inclusion maps which are not retracts.

3. Given two topological spaces X and Y, let $p : X \times Y \to X$ and $q : X \times Y \to Y$ denote the respective coordinate projections.

(i) Prove that for every space Z the maps of homotopy classes $p_* : [Z, X \times Y] \to [Z,X]$ and $q_* : [Z, X \times Y] \to [Z,Y]$ define a 1–1 correspondence θ_Z from $[Z, X \times Y]$ to $[Z,X] \times [Z,Y]$, and if $g : W \to Z$ is a continuous mapping then we have the following commutative diagram (in other words, the value of the composite function is the same for both paths):

$$\begin{array}{ccc}
[Z, X \times Y] & \xrightarrow{\theta_Z} & [Z,X] \times [Z,Y] \\
\big\downarrow g^* & & \big\downarrow g^* \\
[W, X \times Y] & \xrightarrow{\theta_W} & [W,X] \times [W,Y]
\end{array}$$

(ii) Suppose now that X is a topological group, and for each space Y define a binary operation on $[Y,X]$ by composing the isomorphism $[Y,X] \times [Y,X] \cong [Y, X \times X]$ with the map $m_* : [Y, X \times X] \to [Y,X]$ induced by the continuous multiplication mapping $m : X \times X \to X$. Prove that this operation defines a group structure on $[Y,X]$ and for each $h : Z \to Y$ the map $h^* : [Y,X] \to [Z,X]$ defines a group homomorphism.

4. Show that if A is a star convex subset of $\mathbb{R}^n$ with "focal point" a_0 (see Exercise VII.2.1 above), then for all pointed spaces (X, x_0) show that

(a) every basepoint preserving map from (A, a_0) to (X, x_0) is basepoint preserving homotopic to the constant map with value x_0,

(b) every basepoint preserving map from (X, x_0) to (A, a_0) is basepoint preserving homotopic to the constant map with value a_0.

5. (i) Suppose that $f : X \to Y$ is a homotopy equivalence of topological spaces. Prove that for every space W the mappings $f_* : [W,X] \to [W,Y]$ and $f^* : [Y,W] \to [X,W]$ are isomorphisms. [*Hint:* Start by choosing a homotopy inverse g to f.]

(ii) Using (i) and Exercise VII.1.1, show that if $f : X \to Y$ is a homotopy equivalence then there is a 1–1 correspondence between the arc components of X and the arc components of Y.

6. Let X be the Cantor Set $\cap_n X_n$, where $X_0 = [0,1]$, X_n is a union of 2^n pairwise disjoint closed intervals of length 3^{-n}, and X_{n+1} is obtained from X_n by removing the open middle third from each interval (this was first mentioned in Section I.4 of the course notes). Prove that X cannot have the homotopy type of an open subset in $\mathbb{R}^n$ for any n. [*Hint:* Given two points $u \neq v \in X$, find U and V be disjoint open subsets containing u and v such that $U \cup V = X$. Why does this imply that every arc component of X consists of a single point? Finally, if a space Y is homotopy equivalent to an open subset in $\mathbb{R}^n$, why must it have only countably many arc components?]

VII.4 : Homotopy types

([**M**], § 58; [**H**], Ch. 0)

Munkres, § 58, pp. 366–367: 1

Hatcher, pp. 18–20: 5, 13

Additional exercises

1. Prove that $A = ([-1,1] \times \{0\}) \cup (\{-1,1\} \times [0,1])$ is a strong deformation retract of $X = [-1,1] \times [0,1]$. [*Hint:* Let p be the point $(0,2)$. For each $v = (x,y) \in X$ show that there is a unique point $a \in A$ such that $a = p + t(v - p)$ for some $t > 0$. Show that t and hence a are continuous functions of v; the formula for $t(v)$ is given by two different expressions depending on whether $2|x| + y \leq 2$ or $2|x| + y \geq 2$.]

2. Let X be a topological space, and let A be a closed subset of X. Prove that there is a space X', a homotopy equivalence $h : X \to X'$ and a homeomorphism h_0 from A onto a closed subset $A' \subset X'$ such that (a) if j denotes the inclusion of A', then $j \circ h_0$ is homotopic to $h|A$, (b) there is an open neighborhood U of A' in X' such that A' is a strong deformation retract of U. [*Hint:* Let $X' \subset X \times [0,1]$ be the subset $X \times \{0\} \cup A \times [0,1]$, and let $A' = A \times \{1\}$]. — This is a special case of a construction called a **mapping cylinder.**

3. Let $\{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in A}$ be an indexed family of homotopy equivalences of topological spaces. Prove that the product map

$$\prod_{\alpha \in A} f_\alpha : \prod_{\alpha \in A} X_\alpha \quad \longrightarrow \quad \prod_{\alpha \in A} Y_\alpha$$

(as defined in Section II.4 of the course notes) is also a homotopy equivalence. [*Hint:* Recall from Section II.4 that a mapping into a product space is continuous if and only if its coordinate projections are continuous, and use this to construct the required homotopies.]

4. (i) Suppose that a topological space X is equal to $A \cup F$ where A and F are closed subsets, and let $B = A \cap F$. Prove that if B is a strong deformation retract of F, then A is a strong deformation retract of X.

(ii) Suppose that a topological space X is a union of two closed subsets $F_1 \cup F_2$, and let $C = F_1 \cap F_2$. Prove that if C is a strong deformation retract of both F_1 and F_2, then C is also a strong deformation retract of X.

VIII. The fundamental group

VIII.1 : Definitions and basic properties

([**M**], § 52; [**H**], § 1.1)

Munkres, § 52, pp. 334–335: 2, 4, 7

Hatcher, pp. 38–40: 10, 13

Additional exercises

1.* The notation $\pi_1(X,x)$ strongly suggests that one can also define objects $\pi_k(X,x)$ for other values of k. This exercise gives one approach to doing so. To simplify the discussion we shall assume that X is a metric space; everything can be done more generally provided we use more sophisticated results about topological structures on spaces of continuous functions (as in Unit IV of the notes or Section 46 of Munkres).

Given a point space (X,x), let $\Omega(X,x)$ (the **loop space** of (X,x)) denote the space of closed curves in X, viewed as a metric space using the uniform metric:

$$\mathbf{d}(\gamma_1,\gamma_2) \quad = \quad \max_{t\in[0,1]} \mathbf{d}_X\big(\,\gamma_1(t),\gamma_2(t)\,\big)$$

We take the constant curve C_x whose value is x to be a basepoint for $\Omega(X,x)$. Since the loop space of a pointed space is also a pointed space, we can repeat this construction to define a sequence of iterated loops spaces $\Omega^k(X,x)$ for all $k\geq 0$, and if n is a positive integer then the $n^{\rm th}$ **homotopy group** $\pi_n(X,x)$ is defined to be the fundamental group of $\Omega^{n-1}(X,x)$ with its standard basepoint.

(i) Prove that the concatenation construction defines a continuous mapping from $\Omega(X,x)\times\Omega(X,x)$ to $\Omega(X,x)$ and the maps $J_1,J_2:\Omega(X,x)\to\Omega(X,x)$, which send a closed curve γ to $\gamma+C_x$ and $C_x+\gamma$ respectively, are homotopic to the identity.

(ii) Prove that $\pi_1(\Omega(X,x),C_x)$ is abelian. [*Hint:* Imitate the argument in the assigned Exercise 7 from Munkres, using the conclusions of (i) as a weak substitute for the multiplication on a topological group and the identity $g\cdot 1 = g = 1\cdot g$.] — Why does this imply that $\pi_n(X,x)$ is an **abelian** group if $n\geq 2$?

(iii) Assume now that we have a second pointed metric space (Y,y) and a *uniformly* continuous map $f:(X,x)\to(Y,y)$. Prove that the mapping $\Omega(f):\Omega(f)(X,x)\to\Omega(f)(Y,y)$, which sends a closed curve γ to the composite $f\circ\gamma$, is also uniformly continuous. In addition to this, prove that $\Omega(f)$ is compatible with the concatenation operation: $\Omega(\gamma_1+\gamma_2)=\Omega(\gamma_1)+\Omega(\gamma_2)$ for all γ_1 and γ_2.

(iv) Using the preceding parts of this exercise, explain why the $n^{\rm th}$ homotopy group construction extends to a covariant functor from pointed metric spaces and uniformly continuous maps to abelian groups if $n\geq 2$, and to a covariant functor from pointed compact metric spaces and continuous maps to abelian groups if $n\geq 2$.

Note. A pointed topological space (X,e) which has a continuous binary operation $m: X\times X\to X$ such that $m|X\times\{e\}$ and $m|\{e\}\times X$ are both homotopic to the identity on X is called an H**-space** or a **Hopf space**. Topological spaces with this additional structure are named for Heinz Hopf (1894–1971), who made several fundamental contributions to algebraic topology and modern differential geometry (and was not related to Eberhard Hopf (1902–1983), who made important contributions to a wide very range of topics in analysis).

2. Let (X, x) be a pointed space. For many purposes it is also convenient to define the pointed set $\pi_0(X, x)$ to be the the set $[(S^0, 1), (X, x)]$ of all base point preserving homotopy classes of continuous mappints $(S^0, 1) \to (X, x)$, with the constant map (whose value everywhere is x) as its basepoint. Explain why $\pi_0(X, x)$ is naturally in 1–1 correspondence with the set of arc components in X such that the basepoint corresponds to the arc component of x.

Note. One formal justification for this notation is that there is a natural isomorphism of pointed sets with appropriate binary operations $\pi_1(X, x) \cong \pi_0(\Omega(X, x), C_x)$ for suitable spaces X.

VIII.2 : Important special cases

(Munkres, §§ 53–54, 65, 73; Hatcher, § 1.1)

Munkres, § 58, pp. 366–367: 2*acdfghij* (see the note below), 9*bcde* with the definition of degree given below

Hatcher, pp. 38–40: 3, 16*ac*, 17 with $S^1 \times S^1$ replacing $S^1 \vee S^1$

Note. For Exercise 58.2 in Munkres, the fundamental group is either trivial or cyclic for the examples listed above; it suffices to give correct answers without detailed proofs.

Definition. (For the purposes of this course) If $f : S^1 \to S^1$ is a continuous mapping, then the *degree* of f, written $\deg(f)$ is the integer defined as follows: Let $\omega(t) = \exp 2\pi i\, t$, let $t_0 \in \mathbb{R}$ be such that $p(t_0) = f \circ \omega(0)$ — where $p : \mathbb{R} \to S^1$ is the usual map $p(t) = \exp 2\pi i\, t$, take β to be the unique path lifting of $f \circ \omega$ starting at t_0, and set *deg* (f) equal to the unique integer n such that $\beta(1) = t_0 + n$. This integer exists because $p \circ \beta(1) = p \circ \beta(0)$, which means that $\beta(1) - \beta(0)$ is an integer. Since an arbitrary lifting α of $f \circ \omega$ is given by $\alpha(t) = \beta(t) + m$ for some integer m, it follows that $\deg(f)$ does not depend upon the choice of t_0. — It is recommended that Exercise 0 below be worked before doing the exercise in Munkres with our definition of degree.

Additional exercises

0. Suppose that $f : (S^1, 1) \to (S^1)$ is basepoint preserving. Prove that the induced homomorphism of fundamental groups

$$f_* : \mathbb{Z} \cong \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1) \cong \mathbb{Z}$$

is multiplication by $\deg(f)$.

1. Suppose that we are given continuous mappings $f, g : S^1 \to S^1$, and define $h(z)$ to be their product as complex numbers: $h(z) = f(z) \cdot g(z)$. Prove that $\deg(h)$ is equal to $\deg(f) + \deg(g)$. [*Hint:* Recall that $p(t_1 + t_2) = p(t_1) \cdot p(t_2)$.]

2.* By Corollary VIII.2.5 we know that the endpoint preserving homotopy classes of curves joining pairs of points in a topological space X define a special type of category called a *groupoid* which is called the *fundamental groupoid* of X and will be denoted by $\Pi(X)$. The objects of this category are in 1–1 correspondence with the points of X, and the object corresponding to $x \in X$ is merely $\pi_1(X, x)$.

(i) Suppose that $f : X \to Y$ is a continuous mapping of spaces. Prove that the construction sending an endpoint preserving homotopy class $[\gamma]$ to the endpoint preserving homotopy class $[f \circ \gamma]$ defines a covariant functor of groupoids from $\Pi(X)$ to $\Pi(Y)$.

(*ii*) Let $\mathcal{G}$ be an arbitrary groupoid, and suppose that there is a morphism $\alpha : A \to B$. Prove that $\mathrm{Mor}_{\mathcal{G}}(A, A)$ and $\mathrm{Mor}_{\mathcal{G}}(B, B)$ are groups with respect to the composition operation in $\mathcal{G}$ and that these two groups are isomorphic. [*Hint:* This generalizes something in the course notes.]

(*iii*) Given a category $\mathcal{A}$, let $\mathcal{G}$ consist of all objects in $\mathcal{A}$ together with all morphisms in $\mathcal{A}$ which are isomorphisms. Prove that $\mathcal{G}$ is a groupoid.

For the next two exercises, the k-dimensional torus T^k is the product of k copies of S^1 with itself, the basepoint $e \in T^k$ has all coordinates equal to 1, and we shall use the standard identification J_k of $\pi_1(T^k, e)$ with $\mathbb{Z}^k$ given by the isomorphisms $\pi_1(S^1, 1) \cong \mathbb{Z}$ and $\pi_1(A \times B) \cong \pi_1(A) \times \pi_1(B)$ derived in this section. Also, the map $p_k : \mathbb{R}^k \to T^k$ will be the product of k copies of the covering space mapping $p : \mathbb{R} \to S^1$.

3. (*i*) Let $m : S^1 \times S^1 \to S^1$ be the multiplication map. Prove that the composite

$$J_1 \circ m_* \circ J_2^{-1} : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$$

is the addition map sending (u, v) to $u + v$. [*Hint:* Let i_1 and i_2 be the slice inclusions mapping S^1 to $S^1 \times \{1\}$ and $\{1\} \times S^1$ respectively. Show that $J_2 \circ i_{1*} \circ J_1^{-1}$ and $J_2 \circ i_{2*} \circ J_1^{-1}$ are the slice inclusions onto $\mathbb{Z} \oplus \{0\}$ and $\{0\} \oplus \mathbb{Z}$ respectively.]

(*ii*) Let $c_1, \cdots, c_k \in \mathbb{Z}$ and define $f : T^k \to S^1$ by the formula

$$f(z_1, \cdots, z_k) \quad = \quad \prod_{j=1}^{k} z_j^{c_j} \ .$$

Prove that $J_1 \circ f_* J_k^{-1}$ sends the standard unit vector $\mathbf{e}_j \in \mathbb{Z}^k \subset \mathbb{R}^k$ (with a 1 in the j^{th} coordinate and zeros elsewhere) to c_j.

4. (*i*) Let A be an $m \times n$ matrix with integral entries, and define $\Phi_A : T^n \to T^m$ by the formula

$$\Phi_A(p_n(\mathbf{v})) \quad = \quad p_n(A\,\mathbf{v})$$

for $\mathbf{v} \in \mathbb{R}^m$; we are identifying $\mathbb{Z}^k$ with the set of all $k \times 1$ column vectors with integral coefficients. The mapping Φ_A is well defined because matrix multiplication is a continuous operation and $p_m(\mathbf{v}) = p_k(\mathbf{w})$ implies that $\mathbf{v} - \mathbf{w}$ lies in $\mathbb{Z}^k$, which in turn implies that

$$A\,\mathbf{v} \ - \ A\,\mathbf{w} \quad = \quad A(\mathbf{v} \ - \ \mathbf{w}) \quad \in \quad \mathbb{Z}^n$$

and hence $p_n(A\,\mathbf{v}) = p_n(A\,\mathbf{w})$. — Prove that the mapping $J_m \circ \Phi_{A*} \circ j_n^{-1}$ from $\mathbb{Z}^n$ to $\mathbb{Z}^m$ sends $\mathbf{b} \in \mathbb{Z}^n$ to $A\,\mathbf{b}$. [*Hint:* It suffices to do this one coordinate at a time. For that case, use the conclusions of the preceding exercise.]

(*ii*) In the setting above, explain why Φ_A is a homeomorphism if and only if $\det A = \pm 1$. [*Hint:* If an integral matrix is invertible such that the inverse has integral entries, why must its determinant be equal to ± 1?]

(*iii*) Explain why the preceding observations lead to the following conclusion: *Every homomorphism φ from $\pi_1(T^n, e)$ to itself has the form f_* for some basepoint preserving self-map of $(T^n, 1)$. Furthermore, we can find a self-homeomorphism f such that $f_* = \varphi$ if and only if φ is an automorphism of $\pi_1(T^n, e)$.*

VIII.3 : Covering spaces

(Munkres, 53; Hatcher, 1.3)

Munkres, § 53, p. 341: 1, 2, 4, 5, 6

Hatcher, pp. 79–82: 2, 3

Additional exercises

1. (i) Let $p : (E, e_0) \to (B, b_0)$ be a covering space projection with X and Y both (arcwise) connected, and suppose that we are given a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & X \\ \downarrow p & & \downarrow f \\ B & \xrightarrow{\varphi} & Y \end{array}$$

such that Φ and φ are homeomorphisms. Prove that f is also a covering space projection.

(ii) Let $q : E \to B$ be a continuous map such that there is an open covering $\mathcal{U}$ of B by sets U_α such that the maps $q_\alpha : q_\alpha^{-1}[U_\alpha] \to U_\alpha$ determined by q are all covering space projections. Prove that q is also a covering space projection.

2. Suppose we are given covering space projections $p_i : E_i \to X$ $(i = 1, 2)$ with a factorization $p : E_1 \to E_2$ such that $p_1 = p_2 \circ p$. Assume that X is locally connected and p is onto. Prove that p is also a covering space projection.

3. Let $p : E \to X$ be a covering space projection, and let $f : Y \to X$ be continuous. Define the *pullback*

$$Y \times_X E := \{(e, y) \in Y \times E |\ f(y) = p(e)\}.$$

Let $p_{(Y,f)} = \mathrm{proj}_Y | Y \times_X E$.

(i) Prove that $p_{(Y,f)}$ is a covering space projection. Also prove that there is a continuous mapping $\varphi : Y \to E$ such that $p \circ \varphi = f$ (a *lifting* of f to E) if and only if there is a map $s : Y \to Y \times_X E$ such that $p_{(Y,f)} s = 1_Y$.

(ii) Suppose also that f is the inclusion of a subspace. Prove that there is a homeomorphism $h : Y \times_X E \to p^{-1}(Y)$ such that $p \circ h = p_{(Y,f)}$.

NOTATION. If the condition in (ii) holds we sometimes denote the covering space over Y by $E|Y$ (in words, E restricted to Y). Note that $E|Y$ is not necessarily connected even if E is connected; for example, if Y is an evenly covered open set and the covering has more than one sheet, then $E|Y$ will be disconnected.

4. Suppose that $p : E \to X$ is a covering space projection and X is totally disconnected (*i.e.*, the topology has a base of sets that are both open and closed). Prove that E is also totally disconnected.

5. (i) Suppose that $p : E \to X$ is a covering space projection where X is connected and second countable, and assume that at each point of X the number of sheets is (finite or) countable. Prove that E is also second countable.

(ii) Under the hypotheses of (i), explain why E is metrizable if X is metrizable. [*Hint:* By an exercise in Munkres we know that X is $\mathbf{T}_3$. Apply the Urysohn Metrization Theorem.]

VIII.4 : Fundamental groups of spheres

(Munkres, § 59)

No exercises for this section.

VIII.5 : Simply connected spaces

(Munkres, § 53)

Additional exercises

1. If $n \geq 3$ is an integer, then the *dihedral group of order* $2n$, written D_{2n} here, is the subgroup of the group O_2 of orthogonal 2×2 matrices generated by the following elements:

(a) Rotation R_k through an angle of $2\pi\, ik/n$, where $1 \leq k \leq n-1$.

(b) Reflection S about the $x-$axis sending (x, y) to $(x, -y)$.

One can check that $S = S^{-1}$ and $SR_kS = R_{n-k}$ for all k, and this implies that every element of the group can be written uniquely in the form $S^i\, R_1^j$ where $(i, j) \in \{0, 1\} \times \{0, \;\cdots\;, n-1\}$, which implies that the group does have order 2n. Note also that the matrices R_1^i determine a normal cyclic subgroup C_n of order n. The purpose of this exercise is to construct examples of spaces whose fundamental groups are isomorphic to D_{2n} for all n.

(i) Construct an action of D_{2n} on S^3 by embedding D_n into the 4×4 orthogonal matrices such that the 2×2 matrix A maps to the block sum

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

where all blocks are 2×2; let $\sigma(A)$ denote the 4×4 matrix obtained in this way. By construction σ is a homomorphism. Now construct an action of D_{2n} on S^2 using the quotient projection $\Delta : D_{2n}/C_n \cong \mathbb{Z}_2 = \{\pm 1\}$ and mapping (A, x) to the scalar product $\Delta(A) \cdot x$. Prove that the diagonal action on $S^2 \times S^3$, sending $(A; x, y)$ to $(\Delta(A) \cdot x, \sigma(A)\, y)$, is a free action of D_{2n} on $S^2 \times S^3$.

(ii) Explain why there is a compact (and Hausdorff and locally arcwise connected) space X whose fundamental group is isomorphic to D_{2n}.

(iii) Let $X = S^2 \times S^3/D_{2n}$ as above. Show that the coordinate projection $S^2 \times S^3 \to S^2$ passes to a well-defined continuous function from $X \to \mathbb{RP}^2$, and that this map induces a nontrivial homomorphism of fundamental groups.

2. There is a natural embedding of $S^1 \subset \mathbb{RP}^2$ given by taking a great circle in S^2 and observing that its ends get identified under the quotient projection $S^2 \to \mathbb{RP}^2$. Prove that this inclusion $S^1 \subset \mathbb{RP}^2$ is not a retract, and generalize the conclusion to all $n \geq 2$.

3. (i) Let $n \geq 2$ and let $p : (X, x_0) \to (\mathbb{RP}^n \times \mathbb{RP}^n, y_0)$ be a covering space projection such that X is connected. Prove that either p is a homeomorphism or else the covering has a (finite) even number of sheets.

(ii) Suppose that X is a connected space satisfying the default hypotheses and $\pi_1(X)$ is finite of odd order. Prove that X has no connected 2-sheeted covering spaces.

VIII.6 : Homotopy of paths and line integrals

(Munkres, § 56)

Additional exercises

1. Let p be an analytic function defined on an open neighborhood of the closed disk of radius $\leq R$ centered at 0 such that p has no zeros on the boundary circle. Prove that p has a zero in the disk if the winding number integral is nonzero.

2. (A version of Rouché's Theorem.) In the situation above, suppose that $|q| < |p|$ for $|z| \leq R$ and that the winding number integral for p is nonzero. Prove that the winding number integral for $p + q$ is also nonzero.

3. Let U be an open connected subset of the complex plane, let γ be a regular piecewise smooth closed curve in U, and let f be an analytic function defined on U. Explain why there are only countably many possible values for the line integral

$$\int_\gamma f(z)\,dz \ .$$

Also, show that if there are only finitely many values for this line integral, then there is only one value. [*Hint:* If there are only finitely many values, explain why the line integral defines a group homomorphism from $\pi_1(U)$ to the additive group $\mathbb{C}$ with a finite image.]

Semi-explicit description of the winding number. For the sake of completeness we shall describe the degree of the curve $\Gamma(p, R)$ in Exercises 1 and 2 more precisely. If f is an analytic function defined on an open neighborhood of the closed disk $|z| \leq R$ which is never zero on the boundary, then basic results in complex analysis imply that the set of zeros for f is isolated. Therefore there are only finitely many zeros of f inside the closed disk, and we shall denote them by $a_1 \ , \cdots \ , a_n$. At each a_j one has a power series expansion of f near a_j in powers of $z - a_j$, and for points near a_j we can factor f as a product $(z - a)^M\, g(z)$ where g is an analytic function defined near a_j which is never zero. The positive integer M is called the order or multiplicity of the zero at a_j and will be denoted by m_j. Results in complex analysis then imply that the degree of $\Gamma(p, R)$ is given by $\sum_j \ m_j$, and it can be thought of as the "algebraic" number of zeros that f has in the open disk. — Given this discussion, one can restate Exercise 2 in the usual version of Rouché's Theorem: In the setting of the exercise, f and $f + g$ have the same algebraic number of zeros in the open disk of radius R.

IX. Computing fundamental groups

IX.1 : Free groups

([**M**], §§ 67–69; [**H**], § 1.2)

Additional exercises

1. Let F be a free group on two generators. Find all subgroups $H \subset F$ such that the index of H in F is equal to 2.

2. Let F be a free group on a set of free generators X, let $Y \subset X$, and let $H \subset F$ be the smallest normal subgroup containing Y. Prove that the quotient group F/H is free, and describe a set of generators explicitly in terms of the given data.

3.* Let F_n be a free group on generators $g_1, \cdots, g_n$, let $\mathbb{Z}^n$ be the free abelian group generated by the standard unit vectors $\mathbf{e}_i$, and let $h : F_n \to \mathbb{Z}^n$ be the homomorphism sending g_i to $\mathbf{e}_i$ for each value of i. Finally, let $\mathbf{Aut}(F_n)$ denote the group of all automorphisms of F_n.

(i) Prove that for every $T \in \mathbf{Aut}(F_n)$ there is a unique homomorphism $\theta(T) : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $h \circ T = \theta(T) \circ h$, and this construction satisfies the identity $\theta(T_1 \circ T_2) = \theta(T_1) \circ \theta(T_2)$.

(ii) In the notation of (i), prove that if $T \in \mathbf{Aut}(F_n)$ then $\theta(T)$ is an automorphism of $\mathbb{Z}^n$. [*Hint:* If T is the identity automorphism, why is $\theta(T)$ also the identity?] — Since the automorphism group of $\mathbb{Z}^n$ is just the group $\mathbf{GL}(n, \mathbb{Z})$ of $n \times n$ matrices which have inverses with integral entries, it follows that the map $T \to \theta(T)$ defines a homomorphism θ_n from $\mathbf{Aut}(F_n)$ into $\mathbf{GL}(n, \mathbb{Z})$.

(iii) Prove that θ_2 is onto. [*Hint:* Note that the group $\mathbf{GL}(2, \mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

by results from the 201 sequence, so it suffices to show that each of these matrices lies in the image of θ_2. There are fairly simple choices of preimages for each of these matrices — describe them explicitly.]

Further remarks. A fundamental result of J. Nielsen implies that θ_2 is an isomorphism, and for $n \geq 3$ one can prove that θ_n is onto using the methods of this exercise, but if $n \geq 3$ the mapping is not 1–1. There is additional information on $\mathbf{Aut}(F_n)$ in the following references. The first contains classical material, and the second discusses more recent work up to about 12 years ago. During the past few decades there has been a great deal of activity devoted to analyzing $\mathbf{Aut}(F_n)$, and it is a very active topic of current research in a subject called *geometric group theory.*

W. Magnus, A. Karrass, and D. Solitar. Combinatorial group theory. Presentations of groups in terms of generators and relations (Reprint of the 1976 Second Edition). *Dover Publications, Mineola NY*, 2004.

K. Vogtmann. Automorphisms of free groups and outer space. Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000). *Geometriæ Dedicata* **94** (2002), 1–31. Available online with color illustrations at

`http://www.math.cornell.edu/~vogtmann/papers/Autosurvey/autosurvey.pdf`

4. (i) Let G be a finite group of order n. Prove that G is isomorphic to a homomorphic image of the free group F_{n-1} on $n-1$ generators. [Hint: The nontrival elements of the group can be listed as $g_1, \cdots g_{n-1}$.]

(ii) If G is a finite group of odd order $2k+1$, show that G is isomorphic to a homomorphic image of a free group on k generators. [*Hint:* If $1 \neq g \in G$, why do we have $g \neq g^{-1}$?]

IX.2 : Sums and pushouts of groups

(Munkres, § 68; Hatcher, § 1.2)

Munkres, § 68, pp. 421: 2, 3

Munkres, § 69, pp. 425: 1, 3, 4

Additional exercises

1. Given three groups G, H and K, use the Universal Mapping Property to show that iterated free products $(G * H) * K$ and $G * (H * K)$ are both isomorphic to the threefold free product $G * H * K$. Similarly, show that $G * H$ is isomorphic to $H * G$.

2. (i) Give examples of groups H_1, H_2 and K such that H_1 is not isomorphic to H_2 but $H_1 * K$ is isomorphic to $H_2 * K$. [*Hint:* Let K be a free group on infinitely many generators, let H_1 be finite but nontrivial, and let $H_2 = H_1 * K$.]

(ii) Give examples of abelian groups H_1, H_2 and K such that H_1 is not isomorphic to H_2 but $H_1 \times K$ is isomorphic to $H_2 \times K$. [*Hint:* Let K be a free abelian group on infinitely many generators, let H_1 be finite but nontrivial, and let $H_2 = H_1 \times K$.]

(iii) Give examples of metric spaces X_1, X_2 and Y such that X_1 is not homeomorphic to X_2 but $X_1 \times Y$ is homeomorphic to $X_2 \times Y$. [*Hint:* Let Y be the weak direct sum $\oplus^\infty \mathbb{R}$ consisting of all infinite sequences of real numbers which are zero for all but finitely many terms, and define a metric like the $\mathbf{d}_2$ metric on $\mathbb{R}^m$; note that if we identify $\mathbb{R}^m$ as the subspace of all points such that $x_k = 0$ for $k > m$, then every pair of points in Y lies in some $\mathbb{R}^m$.]

3. Let F_2 be a free group on $\{x, y\}$, and let $h : F_2 \to F_2$ be the automorphism which switches x and y. Prove that $h \circ h$ is the identity, but if $g \in F_2$ and $h(g) = g$ then $g = 1$. [*Hint:* Look at the standard form for elements of F^2 as monomials in powers of x and y.]

IX.3 : The Seifert – van Kampen Theorem

(Munkres, § 70; Hatcher, § 1.2)

Munkres, § 70, p. 433: 1, 3

Additional exercises

1. Suppose that X is an arcwise connected space such that $X = U \cup V$, where U and V are open, nonempty and arcwise connected and their intersection $U \cap V$ is (nonempty and) arcwise connected. Let $p \in U \cap V$.

(i) Show that if U is simply connected and $\pi_1(V,p)$ is abelian, then $\pi_1(X)$ is also abelian.

(ii) Explain why the same conclusion does not hold if we merely assume that $\pi_1(U,p)$ is abelian. It will suffice to give a counterexample.

2. (i) Suppose that X is an arcwise connected space such that $X = U \cup V$, where U, V and $U \cap V$ are all open, nonempty and arcwise connected. Suppose further that the inclusion induced maps from $\pi_1(U \cap V)$ to both $\pi_1(U)$ and $\pi_1(V)$ are surjective. Prove that the inclusion induced map from $\pi_1(U \cap V)$ to $\pi_1(X)$ is also surjective.

(ii) Suppose that X is an arcwise connected space such that $X = U \cup V$, where U, V and $U \cap V$ are all open, nonempty and arcwise connected. Prove that $\pi_1(X)$ is finitely generated if both $\pi_1(U)$ and $\pi_1(V)$ are.

3. Suppose we are given a finitely presented group G with generators x and y and relations $x^3\, y^{-2}$, $x^2\, y\, x^{-2}\, y^{-1}$ and $y^3\, x\, y^{-3}\, x^{-1}$ (hence $x^3 = y^2$, and this element lies in the center because it commutes with a set of generators).

(i) If $[G,G]$ is the commutator subgroup of G, show that the abelianization $G/[G,G]$ is infinite cyclic.

(ii) Let N be the normal subgroup which is normally generated by $x\, y^{-1}$. Prove that $N = G$. [*Hint:* Compare the images of x and y in G/N and recall that $x^3 = y^2$.]

4. Suppose that the topological space X is the union of the arcwise connected open subspaces U and V such that $U \cap V$ is (nonempty and) arcwise connected, where all these spaces have the same base point. Assume further that the associated map of fundamental groups from $\pi_1(U \cap V)$ to $\pi_1(U)$ is onto and the associated map of fundamental groups from $\pi_1(U \cap V)$ to $\pi_1(V)$ is an isomorphism. Prove that the associated map from $\pi_1(U)$ to $\pi_1(X)$ is also an isomorphism.

5. Let $p_1, p_2 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ denote projection onto the first and second coordinates respectively. Prove that the pushout of p_1 and p_2 is trivial.

6. Suppose that (X,x) and (Y,y) are pointed and arcwise connected (Hausdorff) spaces. Let (M,m) be the space formed from $X \amalg [0,1] \amalg Y$ by identifying x with 0 and y with 1, taking the midpoint of $[0,1]$ as the basepoint. Prove that $\pi_1(M,m)$ is isomorphic to the free product of $\pi_1(X,x)$ and $\pi_1(Y,y)$. [*Hint:* Let $U = M - Y$ and $V = M - X$. Why are their respective fundamental groups isomorphic to $\pi_1(X,x)$ and $\pi_1(Y,y)$ respectively?]

IX.4 : Examples and computations

(Munkres, §§ 59, 71–72; Hatcher, Ch. 0, § 1.2)

Additional exercises

1. Let X be the space formed from $S^2 \amalg [-1,1]$ by identifying the endpoints of $[-1,1]$ with the north and south poles of S^2 respectively; explicitly, if $\varepsilon = \pm 1$, then $\varepsilon \in [-1,1]$ is identified with $(0,0,\varepsilon) \in S^2$. The goal of this exercise is to compute the fundamental group of X.

(i) Show that X is homeomorphic to the subspace of $\mathbb{R}^4$ given by $S^2 \cup A$, where we identify $S^2 \subset \mathbb{R}^3$ with $S^2 \times \{0\} \subset \mathbb{R}^4$, and A is the image of the curve $\alpha(t) = (0,0,\cos\pi\, t, \sin\pi\, t)$ for $t \in [0,1]$.

(ii) If $Y = A \cup (D^3 \times \{0\}) \subset \mathbb{R}^4$, show that the induced map of fundamental groups $\pi_1(X) \to \pi_1(Y)$ is an isomorphism.

(iii) Let $B \subset \mathbb{R}^4$ be the closed segment joining $(0,0,1,0)$ to $(0,0,-1,0)$. Prove that B is a strong deformation retract of D^3 and $A \cup B$ is a strong deformation retract of Y. Using this, prove that $\pi_1(X) \cong \mathbb{Z}$.

2. Let $X \subset \mathbb{R}^3$ be the union of S^2 with $D^2 \times \{0\}$. Prove that X is simply connected. [*Hint:* Explain why X is formed by regularly attaching a 2-cell to S^2 as in Proposition IX.4.2.]

3.* The goal of this exercise is to construct a topological space X whose fundamental group is isomorphic to the additive group $\mathbb{Q}$ of rational numbers. (By a result of Shelah cited in the course notes, we cannot find a compact metric space X with this property, but the construction will yield a locally compact metrizable space which is second countable.)

(i) This step is purely algebraic but motivated by the Compact Generation Property of the fundamental group (Proposition VIII.1.12). — Suppose that $\{d_k\}$ is a sequence of positive integers such that every nonzero integer m divides some product $d_1 \cdots d_k$ (for example, take d_k to be the product of the first k positive prime numbers), and consider the diagram

$$A_1 \xrightarrow{h_1} A_2 \to \cdots \to A_k \xrightarrow{h_k} A_{k+1} \to \cdots$$

where each A_k is isomorphic to $\mathbb{Z}$ and the map $h_k : A_k \to A_{k+1}$ is multiplication by d_k. Assume further that we have a group G and homomorphisms $g_k : A_k \to G$ such that (1) $g_k = g_{k+1} \circ h_k$ for all k, (2) the group G is equal to the union of the images of the homomorphisms g_k, (3) if $g_k(x)$ is the trivial element of G, then x lies in the kernel of some composite $A_k \to \cdots \to A_{k+M}$. Prove that each g_k is injective and G is isomorphic to (the additive group) $\mathbb{Q}$. [*Hint:* Let $c_1 = 1$, and for $k > 1$ let $c_k = d_1 \cdots d_{k-1}$, so that $\mathbb{Q}$ is an increasing union of the subgroups $c_k^{-1}\mathbb{Z}$. Use mathematical induction to define a compatible sequence of homomorphisms φ_k on these subgroups, set φ equal to the limit of these homomorphisms as $k \to \infty$, and prove that φ is an isomorphism. To prove the injectivity of the mappings g_k, use the fact that the composites $A_k \to A_{k+M}$ are injective, and deduce the injectivity of φ from this.]

(ii) For each integer d, define a continuous mapping $j[d] : S^1 \to S^1 \times D^2$ by $j[d](z) = (z^d, z)$. Verify that each $j[d]$ is 1–1, and if $\rho : S^1 \times D^2 \to S^1$ is projection onto the first coordinate then ρ is a homotopy equivalence such that $\rho \circ j[d]$ has degree d.

(iii) Now take a sequence of integers $\{d_k\}$ as in (i), and define S_k to be the space constructed out of $S^1 \times [k, k+1] \amalg S^1 \times D^2 \times \{k+1\}$ by identifying $S^1 \times \{k+1\}$ in the first piece with $(\mathrm{Image}\, j[d_k]) \times \{k+1\}$ in the second. Next, construct a space E out of $\coprod_k S_k$ by identifying $S^1 \times \{\mathbf{0}\} \times \{k+1\} \subset S^1 \times D^2 \times \{k+1\} \subset S_k$ with $S^1 \times \{k+1\} \subset S_{k+1}$ for all k, and let E_k be the union of the images of $S_1, \cdots, S_k$. Show that there is a continuous function $\lambda : E \to [0, \infty)$ such that the inverse image of $[k, k+1]$ is the image of S_k for each k, and show that the resulting diagram of fundamental group homomorphisms

$$\pi_1(E_1) \longrightarrow \pi_1(E_2) \to \cdots \to \pi_1(E_k) \longrightarrow \pi_1(E_{k+1}) \to \cdots$$

is the algebraic diagram considered in (i).

(iv) Show each subset E_k is compact and that every compact subset of E is contained in some E_k. Also verify that E is a locally arcwise connected, locally compact, Hausdorff, arcwise connected and second countable space.

(v) Use the conclusions from the preceding parts of this exercise to prove that $\pi_1(E)$ is isomorphic to $\mathbb{Q}$.

(vi) Let S be a nonempty set of positive prime numbers, and let $S^{-1}\mathbb{Z}$ denote the subring of the rationals consisting of all fractions expressible as p/q where q is a monomial in the elements of S. Explain how one can modify the preceding construction to obtain a space X such that $\pi_1(X) \cong S^{-1}\mathbb{Z}$.