Addendum to Section VIII.3

The purpose of this note is to prove some further results along the lines of Munkres, Exercise 54.6.

Local properties and covering space projections

The cited exercise in Munkres shows that if $p : X \to Y$ is a covering space projection and Y has certain topological properties, then X also has these properties. There are numerous other results of this type. Here is a typical example:

THEOREM 5. *Suppose that $p : X \to Y$ is a covering space projection such that Y is locally connected. Then X is also locally connected.*

The following lemma is a key step in the proof:

LEMMA 6. *Let X be a topological space, and suppose that X has an open covering $\mathcal{U} = \{U_\alpha\}$ such that each U_α is locally connected. Then X is locally connected.*

Proof of Lemma 6. Let $x \in X$, suppose that $x \in U_\alpha$, and let V be an open neighorhood of x in X. Then $V \cap U_\alpha$ is an open neighborhood of X in both U_α and X, and since U_α is locally connected there is an open subset $W \subset U_\alpha$ such that W is connected and $x \in W$. Since W is open in U_α and the latter is open in X, it follows that W is also open in X. Therefore there is a connected neighborhood base for x in X; since x was arbitrary, this means that X is locally connected.■

Proof of Theorem 5. We claim that there is an open covering $\mathcal{W}$ of Y by open sets which are evenly covered and connected. For each y there is some evenly covered open neighborhood U_y, and since Y is locally connected there is some open set V_y such that $v \in V_y \subset U_y$ and V_y is connected. The sets V_y are evenly covered because a subset of an evenly covered subset is also evenly covered, and the sets V_y yield the desired open covering $\mathcal{W}$. Note that each of the open sets V_y is locally connected by the definition of that concept.

Consider the open covering $p^{-1}[\mathcal{W}]$ of X defined by the inverse images $p^{-1}[V_y]$. These sets have open coverings by pairwise disjoint open subsets, each of which is homeomorphic to V_y. Therefore the lemma implies that each open subset $p^{-1}[V_y]$ is locally connected. If we now apply the lemma to $p^{-1}[\mathcal{W}]$ we see that X must also be locally connected.■

There are many similar results with other hypotheses of the same general type. Here are two examples:

COROLLARY 7. *A similar conclusion applies if we replace* locally connected *with* locally arcwise connected *or* locally contractible *(every point has a neighborhood base of contractible open subsets).*

The proof of Theorem 5 goes through in each case, and the only necessary change is to replace "locally connected" with "locally arcwise connected" or "locally contractible" throughout the discussion.■