Mathematics 205B, Winter 2014, Examination 2

Answer Key

Unless explicitly stated otherwise, all spaces are assumed to be **Hausdorff** and **locally arcwise connected**.

1. [*25 points*] Suppose that $f : A \to B$ and $g : B \to C$ define an exact sequence and $f' : A' \to B'$ and $g' : B' \to C'$ define another exact sequence. Prove that the direct sums $f \oplus f' : A \oplus A' \to B \oplus B'$ and $g \oplus g' : B \oplus B' \to C \oplus C'$ also define an exact sequence.

SOLUTION:

The simplest way to prove this is to observe that if $h : G \to K$ and $h' : G' \to K'$ are homomorphisms of abelian groups, then

$$\mathrm{Kernel}\,(h \oplus h') \quad \cong \quad (\mathrm{Kernel}\,h) \oplus (\mathrm{Kernel}\,h')\,, \qquad \mathrm{Image}\,(h \oplus h') \quad \cong \quad (\mathrm{Image}\,h) \oplus (\mathrm{Image}\,h')\,.$$

These follow because $h \oplus h'(u, v) = \big(\,h(u), h'(v)\,\big)$.

With these identities at our disposal, the proof of exactness is a consequence of the following chain of identities, in which the middle equality is just the exactness assumption in the problem:

$$\mathrm{Kernel}\,(g \oplus g') \quad \cong \quad (\mathrm{Kernel}\,g) \oplus (\mathrm{Kernel}\,g') \quad = \quad (\mathrm{Image}\,f) \oplus (\mathrm{Image}\,f') \quad = \quad \mathrm{Image}\,(f \oplus f')$$

2. [*25 points*] As in the take-home assignment, suppose that we are given a simplex in $\mathbb{R}^2$ with vertices **a**, **b** and **c**, and identify $\mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$. Let $\mathbf{x}_\pm \in \mathbb{R}^3$ denote the point $(0,\,0,\,\pm 1)$. Consider the simplicial complex K which is the union of the two 3-simplices $\mathbf{x}_\pm\,\mathbf{a}\,\mathbf{b}\,\mathbf{c}$, and assume that the vertices have the linear ordering $\mathbf{x}_-$, $\mathbf{x}_+$, **a**, **b**, **c**. Find a 3-chain

$$u\,\mathbf{x}_+\,\mathbf{a}\,\mathbf{b}\,\mathbf{c} \;+\; v\,\mathbf{x}_-\,\mathbf{a}\,\mathbf{b}\,\mathbf{c} \quad \in \quad C_3(\mathbf{K}^\omega)$$

whose boundary is a linear combination of all 2-simplices except **abc**. [*Hint:* There is such a chain with $|u|, |v| \leq 1$.]

SOLUTION:

Let c be the displayed 3-dimensional chain. We want to find u and v such that $d_3(c)$ has no terms involving **abc**. The easiest way to address this is to compute the boundary semi-explictly; more correctly, it suffices to compute the terms in the boundary involving **abc**.

If we do this, we find that $d_3(c) = (u+v)\,\mathbf{abc} \;+\; T$, where T is a "trash term" which is a linear combination of the other standard free generators for $C_2(\mathbf{K}^\omega)$ — namely, the symbols corresponding to 2-simplices in **K** which have $\mathbf{x}_-$ or $\mathbf{x}_+$ as a vertex. Therefore the condition in the problem is equivalent to $u = -v$, and one specific example is given by

$$\mathbf{x}_+\,\mathbf{a}\,\mathbf{b}\,\mathbf{c} \;-\; \mathbf{x}_-\,\mathbf{a}\,\mathbf{b}\,\mathbf{c}\,. \blacksquare$$

3. [*20 points*] State the Mayer-Vietoris axiom for singular homology as it applies to a topological space X which is decomposed as the union of two open subsets U and V.

SOLUTION:

This property is stated explicitly in the course notes.

4. [*30 points*] Let $m \geq 2$ be an integer, let $U \subset \mathbb{R}^m$ (m, not n) be a nonempty open subset, and let $n < m$ be another integer. Prove that there is no 1–1 continuous mapping $f : U \to \mathbb{R}^n$. [*Hints:* Let j be the usual inclusion of $\mathbb{R}^n$ to $\mathbb{R}^m$ obtained by adjoining $m - n$ zero coordinates. What does Invariance of Domain imply about $j \circ f$? Observe that the image of j has an empty interior.]

SOLUTION:

Follow the hint, and assume that there is a 1–1 continuous mapping $f : U \to \mathbb{R}^n$. Since f and j are both 1–1 the composite $j \circ f : U \to \mathbb{R}^m$ is a continuous 1–1 mapping from an open subset of $\mathbb{R}^m$ into $\mathbb{R}^m$. Therefore by Invariance of Domain the mapping $j \circ f$ is open.

On the other hand, the image of this map is contained in the image of j, and the latter has empty interior (in fact, its complement is dense, or alternatively given a point $\mathbf{x} \in \mathrm{Image}\, j$ no point of the form $\mathbf{x} + t\,\mathbf{e}_m$ $t \neq 0$ lies in this image, and if the image were open such points would like in it for $|t|$ sufficiently small). Therefore the nonempty set $\mathrm{Image}\, j \circ f$ cannot be open in $\mathbb{R}^m$, which contradicts our earlier conclusion.

The source of this contradiction was the assumption about the existence of a 1–1 continuous mapping $f : U \to \mathbb{R}^n$, and therefore no such mappint can exist.∎