ALL SPACES HAUSDORFF, LOC. ARC. CONNECTED

Practice Problems for Examination 2.

1. Prove that there is no continuous 1-1 mapping from an open subset $U \subseteq \mathbb{R}^m$ into $\mathbb{R}^n$ if $m > n$. [Hint: If $g: U \to \mathbb{R}^n$ is such a map, consider $j \circ g : U \to \mathbb{R}^m$ where $j: \mathbb{R}^n \to \mathbb{R}^m$ is injection onto $\mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^m$. Why is this composite 1-1, and why is its image not open? How does this lead to a contradiction?]

2. Let K be a simplicial complex such that $K = K_0 \cup A$ where A is an n-simplex and $K_0 \cap A$ is the union of all but one of the maximal faces. Prove that the map induced by inclusion $H_*(K_0) \longrightarrow H_*(K)$ is an isomorphism. [Hint: Why is $K_0 \cap A$ star shaped?]

3. Let $U, V, W \subseteq \mathbb{R}^n$ $(n \geq 2)$ be open and convex. Prove that $H_q(U \cup V \cup W) = 0$ if $q \geq 2$ and give an example where $H_1 \neq 0$. [Hint: The intersection of convex sets is convex. First find the homology of $U \cup V$, $U \cup W$, $V \cup W$. To get a counterexample, thicken up the graph given by the edges of a triangle.]

4. We have proved results implying that
$$H_n(S^n \times S^n \times S^n) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$
Given a homeomorphism $f: S^n \times S^n \times S^n \circlearrowleft$, Let A be the 3×3 matrix determined by f_*. Why is $\det A = \pm 1$?

5. Let X and Y be nonempty spaces with $p \in X$, $q \in Y$. Define $i_1 : X \to X \times Y$ $i_1(x) = (x, q)$
$i_2 : Y \to X \times Y$ $i_2(y) = (p, y)$.

Prove that i_{1*} and i_{2*} are 1-1, and their images' intersection $= \{0\}$. Hence $H_q(X \times Y)$ contains a copy of $H_q(X) \oplus H_q(Y)$.

[Hint: The maps i_1, i_2 are retracts with one sided inverses = projections on X & Y resp.
To show the intersection is 0, show that
$\pi_{X*} i_{2*} = 0$, $\pi_{Y*} i_{1*} = 0$
and apply these identities.]

6. Suppose we have a short exact sequence of chain complexes

$$0 \to C_* \to D_* \to (D/C)_* \to 0$$

as follows, with zeros in all other dimensions

$$\begin{tikzcd}
(q+1) & 0 \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & A \arrow[r, "="] \arrow[d, "f"] & A \arrow[r] \arrow[d, "0"] & 0 \\
(q) & 0 \arrow[r] & B \arrow[r, "="] & B \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Show that $H_{q+1}(D/C) \cong A$ and $H_q(C) \cong B$ such that the connecting map

$$\partial : H_{q+1}(D/C) \longrightarrow H_q(C)$$

corresponds to $f : A \to B$.

7. Suppose $A \subseteq B \subseteq X$ where A is a strong deformation retract of both B and X. Prove that the inclusion map induces isomorphisms $H_*(B) \to H_*(X)$.

Give an example to show B is not necessarily a deformation retract of X.

[Hint: There are examples where B is open and dense in X.]

8. Let X_0 be the cube boundary for
$[0,1]\times[0,1]\times[0,1]$, so that X is homeomorphic to S^2. Let X_2 be the graph of line segments joining the centers of squares on opposite sides, so that X_2 consists of the line segments joining

$(\frac{1}{2},\frac{1}{2},0)$ to $(\frac{1}{2},\frac{1}{2},1)$
$(\frac{1}{2},0,\frac{1}{2})$ to $(\frac{1}{2},1,\frac{1}{2})$
$(0,\frac{1}{2},\frac{1}{2})$ to $(1,\frac{1}{2},\frac{1}{2})$.

Draw a picture! Note these meet at $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$.

Assume there is a simplicial decomposition of $X_0 \cup X_1$ such that X_0 + X_1 are subcomplexes. Using this, compute the homology groups of $X = X_0 \cup X_1$.

9. Prove that there is no 1-1 continuous mapping $S^2 \to T^2 (= S^1 \times S^1)$ or in the opposite direction $T^2 \to S^2$. [<u>Hint</u>: These are 2-dim topological manifolds as defined in the course materials. Use Invariance of Domain plus compactness.]

10. Let $p : E \to B$ be a basepoint preserving covering map where E is arcwise connected ($\Rightarrow B$ is too). Prove that E is simply connected if p is basepoint preservingly homotopic to a constant map.

<u>More suggestions.</u>

Make sure you know how to use long exact homology sequences and Mayer–Vietoris sequences to compute homology.

Make sure you know the "axioms" for singular homology.

There will be nothing on Section VII.4