TAKE-HOME SOLUTIONS

1 Start by writing out the Mayer-Vietoris seq:

$$\cdots \xrightarrow{\Delta} H_q(K_1 \cap K_2) \xrightarrow{(i_{1*}, -i_{2*})} \begin{matrix} H_q(K_1) \\ \oplus \\ H_q(K_2) \end{matrix} \xrightarrow{j_{1*}+j_{2*}} H_q(K) \xrightarrow{\Delta} H_{q-1}(K_1 \cap K_2) \cdots$$

RECALL $K_1 \cap K_2$ IS STAR-SHAPED

Suppose that $q \geq 2$: Then $H_q(K_1 \cap K_2) = 0 = H_{q-1}(K_1 \cap K_2)$, so the exact sequence yields

$$0 \longrightarrow \begin{matrix} H_q(K_1) \\ \oplus \\ H_q(K_2) \end{matrix} \xrightarrow[j_{1*}+j_{2*}]{\varphi =} H_q(K) \longrightarrow 0$$

By exactness, $\operatorname{Ker} \varphi = 0$ and $\operatorname{Image} \varphi = H_q(K)$, so we have an isomorphism if $q \geq 2$.

Suppose that $q = 1$: Then we have

$$0 = H_1(K_1 \cap K_2) \longrightarrow \begin{matrix} H_1(K_1) \\ \oplus \\ H_1(K_2) \end{matrix} \xrightarrow{\varphi} H_1(K) \xrightarrow{\Delta} H_0(K_1 \cap K_2) \rightarrow \begin{matrix} H_0(K_1) \\ \oplus \\ H_0(K_2) \end{matrix}$$

$$\mathbb{Z} \xrightarrow{\alpha} \begin{matrix} \mathbb{Z} \\ \oplus \\ \mathbb{Z} \end{matrix}$$

Each of the 0-dim. homology groups is generated by the class $[v]$ of the vertex v s.t. $K_1 \cap K_2$ is star-shaped with respect to v, and we have $\alpha([v]) = ([v], -[v])$.

Hence α is 1-1. By exactness,
Image Δ = Kernel $\alpha = \{0\}$, so $\Delta = 0$.

Now $0 = H_1(K_1 \cap K_2) \Rightarrow$ once again, φ is 1-1.
But now Image φ = Kernel Δ by exactness, and since $\Delta = 0$ this means φ is also onto.
Hence φ is also an isomorphism if $q = 1$.

<u>Suppose that $q = 0$:</u> As in the preceding discussion, $H_0(K_1 \cap K_2) \cong \mathbb{Z}$ but

$$H_0(K_1) \oplus H_0(K_2) \cong \mathbb{Z} \oplus \mathbb{Z}. \blacksquare$$

2. Consider the last part of the Mayer-Vietoris sequence

$$\begin{matrix} H_1(K_1) \\ \oplus \\ H_1(K_2) \end{matrix} \longrightarrow H_1(K) \xrightarrow{\Delta} H_0(K_1 \cap K_2) \xrightarrow{\alpha} \begin{matrix} H_0(K_1) \\ \oplus \\ H_0(K_2) \\ \mathbb{Z} \oplus \mathbb{Z} \end{matrix}$$

The images of $H_1(K_1)$ and $H_1(K_2)$ generate $H_1(K) \Longleftrightarrow \Delta = 0 \Longleftrightarrow \alpha$ is 1-1 (by exactness). Recall that the map $H_0(K_1 \cap K_2) \longrightarrow H_0(K_i)$ is defined as follows. is free abelian with generators given by eq. classes of vertices in the same (arc) component of $K_1 \cap K_2$, and the map in H_0 takes a free generator associated to a vertex of $K_1 \cap K_2$ into its counterpart in K_i. If K_1 & K_2 are arc wise connected, there is only one free generator. Hence if v is a vertex of $K_1 \cap K_2$, then $\alpha([v]_{K_1 \cap K_2})$ is equal to

$([v]_{K_1}, -[v]_{K_2})$. Hence $\boxed{\text{rank}(\alpha) = 1}$

By algebra, Kernel α = Image Δ is free abelian on m generators, where $m+1 = \#$ of components in $K_1 \cap K_2$.

$K_1 \cap K_2$ connected $\Rightarrow m+1 = 1 \Rightarrow m = 0 \Rightarrow$
$\Delta = 0 \Rightarrow H_1(K)$ gen by $H_1(K_1)$ & $H_1(K_2)$.

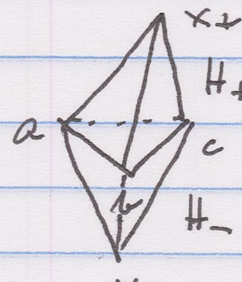
$K_1 \cap K_2$ has $m+1 \geq 2$ components $\Rightarrow \Delta \neq 0$
$\Rightarrow$ Image $H_1(K_1) \oplus H_1(K_2)$ is a subgroup of infinite index in $H_1(K)$. ∎

3. First compute L_0, using the hint.

$$L_0 = H_+ \cup H_-, \text{ where } H_\pm \text{ is starshaped.}$$

Use the M-V sequence, noting that

$H_+ \cap H_- = ab \cup bc \cup ac$ (bdy of abc)



L_0

star shaped

$$0 = \overset{0}{H_2(H_+ \cap H_-)} \longrightarrow \overset{0}{H_2(H_+)} \oplus \overset{0}{H_2(H_-)} \longrightarrow \underset{?}{H_2(L_0)} \xrightarrow{\Delta_2}$$

$$\underset{\mathbb{Z}}{H_1(H_+ \cap H_-)} \longrightarrow \overset{0}{H_1(H_+)} \oplus \overset{0}{H_1(H_-)} \longrightarrow \underset{?}{H_1(L_0)} \xrightarrow{\Delta_1}$$

$$H_0(H_+ \cap H_-) \longrightarrow H_0(H_+) \oplus H_0(H_-) \longrightarrow H_0(\underbrace{H_+ \cup H_-}_{= L_0})$$

$$\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z}$$

$$[v] \qquad ([v], -[v])$$

$\Rightarrow \alpha$ is 1-1.

$\mathbb{Z}$ since L_0 is connected.

By exactness, $\Delta_1 = 0$, so

$$0 \longrightarrow H_1(L_0) \xrightarrow{\Delta_1 = 0} \text{ exact } \Longrightarrow$$

$H_1(L_0) = 0$. Also $0 \longrightarrow H_2(L_0) \xrightarrow{\Delta_2} H_1(H_+ \cap H_-) \to 0$ exact $\Rightarrow \Delta_2$ is an isomorphism (Image Δ_2 = everything, kernel $= \{0\}$), so $H_2(L_0) \cong \mathbb{Z}$.

To summarize, $H_q(L_0) = \begin{cases} \mathbb{Z} & q = 0, 2 \\ 0 & \text{otherwise} \end{cases}$

<u>Now turn to L_1</u> $\quad L_1 = L_0 \cup A$, $A = abc$.

$L_0 \cap A = \Gamma = \text{bdy } A$. We then get the sequence

$$0 = H_2(L_0 \cap A) \longrightarrow \begin{matrix} H_2(L_0) \\ \oplus \\ H_2(A) \end{matrix} \longrightarrow H_2(L_1) \longrightarrow H_1(L_0 \cap A)$$

$$\mathbb{Z} \oplus 0 \qquad ? \qquad \mathbb{Z}$$

$$H_1(L_0) \oplus H_1(A) \longrightarrow H_1(L_1) \xrightarrow{\Delta_1} H_0(L_0 \cap A) \longrightarrow \begin{matrix} H_0(L_0) \\ \oplus \\ H_0(A) \end{matrix}$$

$$0 \oplus 0 \qquad ? \qquad \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z}$$

As before, α is 1-1, so $\Delta_1 = 0$, so $H_1(L_0) \oplus H_1(A) \longrightarrow H_1(L_1)$ is onto, which means $H_1(L_1) = 0$.

We also have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(L_1) \longrightarrow \mathbb{Z} \longrightarrow 0$$

so $H_2(L_1)$ has no nonzero elements of finite order and must be iso to $\mathbb{Z} \oplus \mathbb{Z}$.

Finally, $H_0(L_1) \cong \mathbb{Z}$ by connectedness.

<u>Finally, L_2</u> $\quad L_2 = L_0 \cup E \qquad E = x_+ x_-$

$L_0 \cap E = \{x_+, x_-\}$.

Here is the MV sequence

$$0 = \underset{0 \text{ (0-dim)}}{H_2(L_0 \cap E)} \longrightarrow \underset{\mathbb{Z} \oplus 0}{\begin{matrix} H_2(L_0) \\ \oplus \\ H_2(E) \end{matrix}} \longrightarrow H_2(L_2) \longrightarrow \overset{0}{\underset{\text{(0-dim)}}{H_1(L_0 \cap E)}}$$

$$\longrightarrow \underset{0 \oplus 0}{H_1(L_0) \oplus H_1(E)} \longrightarrow H_1(L_2) \longrightarrow \underset{\mathbb{Z}+\mathbb{Z}}{H_0(L_0 \cap E)} \longrightarrow \underset{\mathbb{Z} \oplus \mathbb{Z}}{\begin{matrix} H_0(L_0) \\ \oplus \\ H_0(E) \end{matrix}}$$

As before $H_0(L_2) = 0$ by connectedness.

The sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(L_2) \longrightarrow 0 \text{ implies}$$

$H_2(L_2) \cong \mathbb{Z}$. ⟵

Finally, we have the exact sequence

$$0 \longrightarrow H_1(L_2) \xrightarrow{\Delta_1} \underset{\mathbb{Z} \oplus \mathbb{Z}}{\overset{}{H_0(L_0 \cap E)}} \xrightarrow{\alpha} \underset{\mathbb{Z} \oplus \mathbb{Z}}{H_0(L_0) \oplus H_0(E)}$$

As before, $[x_+]$ and $[x_-]$ in $H_0(L_0 \cap E)$ both go to $([v], -[v])$ under α, so Kernel α = Image $\Delta_1 \cong$ all $(c_1, c_2) \in \mathbb{Z} \oplus \mathbb{Z}$ such that $c_1 = -c_2$, so that

$$H_1(L_2) \cong \text{Image } \Delta_1 \cong \mathbb{Z}. \blacksquare$$