SOLUTIONS TO SECOND

TAKE–HOME ASSIGNMENT

Winter 2018

1. $\quad(a)$ The idea is to build the complex by starting with ABC and successively adjoining 2-simplices such that each has one edge in common with the previous 2-simplex in the list. This requires and ordering of the 2-simplices as indicated, and one way of doing so is by the sequence listed in the problem:

$$ACD\ ,\ ADE\ ,\ AEF\ ,\ ACF\ ,\ BCD\ ,\ BDE\ ,\ BEF\ ,\ BCF$$

We then have the following boundary formulas:

$$d(ACD) = CD - AD + AC$$

$$d(ADE) = DE - AE + AD$$

$$d(AEF) = EF - AF + AE$$

$$d(ACF) = CF - AF + AC$$

$$d(BCD) = CD - BD + BC$$

$$d(BDE) = DE - BE + BD$$

$$d(BEF) = EF - BF + BE$$

$$d(BCF) = CF - BF + BC$$

We want to choose the signs for the 2-simplices such that the boundary of the chain is $\pm(CD + DE + EF - AF)$, and we shall do so one at a time. We then have

$$d(ACD+ADE) = AC-AE+CD+DE\ , \quad d(ACD+ADE+AEF) = AC-AF+CD+DE+EF$$

$$d(ACD + ADE + AEF - ACF) = CD + DE + EF - CF\ .$$

A similar identity holds with B replacing A:

$$d(BCD + BDE + BEF - BCF) = CD + DE + EF - CF\ .$$

Therefore the difference is a cycle with the desired properties, and it is given explicitly by

$$ACD + ADE + AEF - ACF - BCD - BDE - BEF + BCF\ .\blacksquare$$

(b) In this case we want to start with the 2-simplex DEF at the top and start adding simplices at the sides until the last step, where we adjoin the bottom simplex. In order to speed things up, we shall adjoin several 2-simplices at a time. Each of the 2-simplices ADE, BEF, CDF has an edge in common with DEF, and the boundary of the union of DEF, ADE, BEF, CDF is a simple circuit with edges AD, AE, BE, BF, CD, DF. We want to choose signs for the 2-simplices so that the boundary is a sum of these 1-simplices with appropriate signs and the coefficient of DEF is

$+1$. Now $d(DEF) = EF - DF + DE$, and in order to cancel the 1-simplices in this expression we need to take the 2-chain $DEF - ADE - BEF + CDF$; the boundary of this chain is equal to $-AD + AE - BE + BF + CD - CF$, corresponding to the boundary edge path $DAEBFCD$.

Suppose now that we start with the bottom 2-simplex ABC instead, attaching the three 2-simplices which share an edge with ABC. These are ABE, BCF and ACD. Since $d(ABC) = BC - AC + AB$, the 2-chain we want is $ABC - ABE - BCF + ACD$, and its boundary is also equal to $-AD + AE - BE + BF + CD - CF$.

The two chains in the preceding paragraph have the same boundary, so their difference is the desired cycle, and it is given explicitly by $DEF - ADE - BEF + CDF - ABC + ABE + BCF - ACD$. One thing to note is that the coefficients of the top and bottom simplices are negatives of each other.■

2. (a) Follow the hint, showing that the pair $(\{1\} \times \mathbb{R}^{n-1}, \{1\} \times \mathbb{R}^{n-1})$ a strong deformation retract of $(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{p\})$, where $p \in \{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n_+$. The first pair has trivial homology in each dimension because $H_q(X, X) = 0$ for every space X (look at the long exact homology sequence for the pair (X, X) to see that the relative groups vanish), so if the assertion about deformation retracts is correct then the relative homology of the larger pair is also trivial. Observe that $\mathbb{R}^n_+$ is homeomorphic to $[0, \infty) \times \mathbb{R}^n$; we shall use this splitting in the argument.

A homotopy inverse ρ of pairs from $(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{p\})$ to $(\{1\} \times \mathbb{R}^{n-1}, \{1\} \times \mathbb{R}^{n-1})$ is given by $\rho(t, v) = (1, v)$. If $j : \{1\} \times \mathbb{R}^{n-1} \to \mathbb{R}^n_+$ is the inclusion, then clearly $\rho \circ j$ is the identity, and $j \circ \rho$ is homotopic to the identity by a vertical straight line homotopy:

$$h(u, x; t) \quad = \quad (t + (1-t)u, x) \ , \quad (u, x) \in \mathbb{R}^n_+ \cong [0, \infty) \times \mathbb{R}^n$$

By construction this is a homotopy equivalence of pairs because it sends the subspace $\mathbb{R}^n_+ - \{p\}$ to itself (verify this!).■

(b) If U and V are open subsets of a (Hausdorff) topological space X, with $y \in U$ and $z \in V$, and $f : U \to V$ is a homeomorphism such that $f(y) = z$, then the local homology groups $H_*(U, U - \{y\})$ and $H_*(V, V - \{z\})$ are isomorphic. By excision we know that these local homology groups are isomorphic to $H_*(X, X - \{y\})$ and $H_*(X, X - \{z\})$ respectively, and the conclusion follows by combining this with the observation in the preceding sentence.■

(c) Correction: h should be f.

If $x \in X$, then by (a) the local homology at x is trivial if and only if the first coordinate is zero. Since homeomorphisms preserve local homology groups, it follows that the groups $H_*(X, X - \{x\})$ are trivial if and only if the groups $H_*(X, X - \{f(x)\})$ are trivial. This translates into a conclusion that the first coordinate of x is zero if and only if the first coordinate of $f(x)$ is zero.■