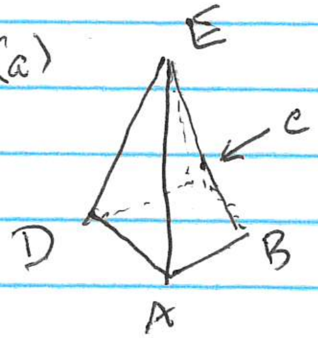
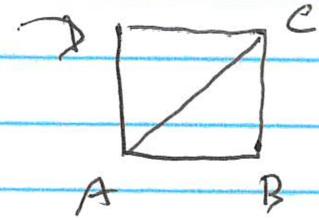
Assignment 2 Solutions

1. (a)



Split the base into ABC and ABD



So the two 3-simplices are ABCE + ABDE.

~~on the base~~.

(b) Now $d(ABCE) = BCE - ACE + ABE - ABC$

$d(ACDE) = CDE - ADE + ACE - ACD$.

We want a linear combination of these in which the inside face terms for ACE cancel.

Just take the sum ABCE + ABDE.

2. (a) We have the following ladder diagram with exact rows extending indefinitely in each direction:

$$\begin{tikzcd}
H_q(B) \arrow[r] \arrow[d, "1"'] & H_q(A) \arrow[r] \arrow[d, "2"'] & H_q(A,B) \arrow[r] \arrow[d, "3"'] & H_{q-1}(B) \arrow[r] \arrow[d, "4"'] & H_{q-1}(A) \arrow[d, "5"'] \\
H_q(V) \arrow[r] & H_q(U) \arrow[r] & H_q(U,V) \arrow[r] & H_{q-1}(V) \arrow[r] & H_{q-1}(U)
\end{tikzcd}$$

Since $A \to U$ is a homotopy equivalence, 2 and 5 are isomorphisms. Since $B \to V$ is too, 1 and 4 are isomorphisms. Therefore 3 is an isomorphism by the Five Lemma.

(b) We have the following commutative diagram induced by pair inclusions

$$\begin{tikzcd}
H_q(A, A\cap B) \arrow[r, "1"] \arrow[d, "2"'] & H_q(A\cup B, B) \arrow[d, "3"'] \\
H_q(U, U\cap V) \arrow[r, "4"'] & H_q(U\cup V, V)
\end{tikzcd}$$

Part (a) shows 2 and 3 are isomorphisms, and excision shows 4 is also an isomorphism. Therefore 1 is also an isomorphism.

3 (a) (WITH CORRECTIONS)

Given $\underset{\underset{A}{\shortparallel}}{\{0\} \times A_0} \subseteq [0,\infty) \times \mathbb{R}^{n-1} \cong \mathbb{R}^n_+$

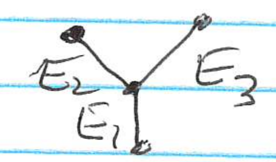
consider the straight line segment
$\gamma(t) = (1-t)y + te_1$ where $0 \le t \le 1$.
We need only check that if $y \in \mathbb{R}^n_+ - A$ then this segment stays inside $\mathbb{R}^n_+ - A$.

Case I If $y \in (0,\infty) \times \mathbb{R}^{n-1}$ this is true since the first coordinate of $\gamma(t)$ is always positive.

Case II $y \in \{0\} \times \mathbb{R}^{n-1}$ In this case the first coordinate is positive if $t > 0$ and this suffices to show $\gamma(t) \in \mathbb{R}^n_+ - A$ if $y \in \mathbb{R}^n_+ - A$.

Therefore $\mathbb{R}^n_+ - A$ has the same homology as a point, and if $A = \{p\}$ this and the exact homology sequence yield $H_q(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{p\}) = 0$.

Since $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \neq 0$ for all $x \in \mathbb{R}^n$, the local homology groups at points of $\mathbb{R}^n_+$ are not all isomorphic to local homology groups for $\mathbb{R}^n$. Therefore $\mathbb{R}^n_+$ and $\mathbb{R}^n$ cannot be homeomorphic.

(b) 

Label the edges as indicated and notice $E_1 \cup E_2 \cong [0,1]$

Write $Y = \underset{\substack{\uparrow \\ E_1 \cup E_2}}{A} \cup \underset{\substack{\uparrow \\ E_3}}{B}$

Then $S^2 - Y = (S^2 - A) \cap (S^2 - B)$

$S^2 - \{p\} = (S^2 - A) \cup (S^2 - B)$.

Thus we have a Mayer-Vietoris exact sequence

$$\tilde{H}_{q+1}(S^2 - \{p\}) \longrightarrow \tilde{H}_q(S^2 - Y) \longrightarrow \begin{matrix} \tilde{H}_q(S^2 - A) \\ \oplus \\ \tilde{H}_q(S^2 - B) \end{matrix}$$

By results from Unit <u>VII</u> we know that the groups on the left and right are zero, so the same is true for the groups in the middle.