# Mathematics 205B, Winter 2012, Take-home assignment 2

ANSWER KEY

**1.** (*a*) Suppose that the simplicial complex $\mathbf{K}$ is a union of two connected subcomplexes $\mathbf{K}_1 \cup \mathbf{K}_2$ where each $\mathbf{K}_i$ is connected, and suppose also that the intersection $\mathbf{K}_1 \cap \mathbf{K}_2$ is starshaped with respect to some vertex $\mathbf{v}$, where $\mathbf{v}$ is minimum in a given linear ordering $\omega$ of the vertices in $\mathbf{K}$. Prove that

$$H_q(\mathbf{K}) \quad \cong \quad H_q(\mathbf{K}_1) \ \oplus \ H_q(\mathbf{K}_2)$$

for all $q > 0$.

SOLUTION: Use the Mayer-Vietoris exact sequence of the decomposition $\mathbf{K} = \mathbf{K}_1 \cup \mathbf{K}_2$:

$$\cdots \to H_{q+1}(\mathbf{K}) \to H_q(\mathbf{K}_1 \cap \mathbf{K}_2) \to H_q(\mathbf{K}_1) \oplus H_q(\mathbf{K}_2) \to H_q(\mathbf{K}) \to \cdots$$

Since the intersection is star shaped, it follows that the $H_q(\mathbf{K}_1 \cap \mathbf{K}_2) = 0$ if $q > 0$, and hence if $q \geq 2$ the sequence consists of pieces having the form

$$0 = H_q(\mathbf{K}_1 \cap \mathbf{K}_2) \to H_q(\mathbf{K}_1) \oplus H_q(\mathbf{K}_2) \to H_q(\mathbf{K}) \to H_{q-1}(\mathbf{K}_1 \cap \mathbf{K}_2) = 0 \ .$$

These exact sequences imply that the map from $H_q(\mathbf{K}_1) \oplus H_q(\mathbf{K}_2)$ to $H_q(\mathbf{K})$ is an isomorphism if $q \geq 2$. We can obtain the same conclusion when $q = 1$ if we know that the map

$$H_0(\mathbf{K}_1 \cap \mathbf{K}_2) \to H_0(\mathbf{K}_1) \oplus H_0(\mathbf{K}_2)$$

is injective since this will imply that $H_1(\mathbf{K}) \to H_0(\mathbf{K}_1 \cap \mathbf{K}_2)$ is once again a trivial map; this in turn implies that the map $H_1(\mathbf{K}_1) \oplus H_1(\mathbf{K}_2)$ to $H_1(\mathbf{K})$ is onto, and the map is also 1–1 since $H_1(\mathbf{K}_1 \cap \mathbf{K}_2) = 0$.

By hypothesis the intersection is connected (because it is starshaped), and therefore $H_0(\mathbf{K}_1 \cap \mathbf{K}_2) = \mathbb{Z}$, and the class of any vertex is a generator. If $\mathbf{L}_1$ is the component of $\mathbf{K}_1$ containing the intersection, then we know that $H_0(\mathbf{K}_1 \cap \mathbf{K}_2) \to H_0(\mathbf{L}_1)$ is an isomorphism and $H_0(\mathbf{L}_1) \to H_0(\mathbf{K}_1)$ is injection onto a direct summand, so this implies that the map $H_0(\mathbf{K}_1 \cap \mathbf{K}_2) \to H_0(\mathbf{K}_1) \oplus H_0(\mathbf{K}_2)$ is automatically injective, and the conclusion about $H_1$ then follows by the preceding argument.■

(*b*) Given a simplicial complex $\mathbf{K}$, prove that it is isomorphic to a subcomplex of a complex $\mathbf{L}$ such that the homology homomorphisms $H_q(\mathbf{K}) \to H_q(\mathbf{L})$ is trivial for all $q > 0$. [*Hint:* Why is $\mathbf{K}$ isomorphic to a subcomplex of some simplex with the same number of vertices?]

SOLUTION: Choose a linear ordering for the $q+1$ vertices of $\mathbf{K}$, and define a map from the underlying space $P$ to the simplex $\Delta_q$ sending a simplex with vertices $\mathbf{v}_{i(0)}, \cdots \mathbf{v}_{i(q)}$ to the face of $\Delta_q$ with vertices $\mathbf{e}_{i(0)}, \cdots \mathbf{e}_{i(q)}$, where $\mathbf{e}_j$ is the standard unit vector (for $j = 0, \ \ldots \ , q$ the $j^{\text{th}}$ coordinate is 1 and the others are zero). This defines an isomorphism from $\mathbf{K}$ to a subcomplex $\mathbf{K}_0$ of $\Delta_q$, and it follows immediately that the inclusion induced mappings from $H_j(\mathbf{K}_0)$ to $H_j(\Delta_q)$ are trivial for $j > 0$.■

**2.** Let $n$ be a positive integer. A *topological $n$-manifold* is a Hausdorff space $M$ such that every point $p \in M$ has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

($a$) Prove that the local homology groups of $M$ at each point $x \in M$ are infinite cyclic in dimension $n$ and zero otherwise. [*Hint:* Use the localization principle for local homology and the fact that $x$ has an open neighborhood homeomorphic to a subset of $\mathbb{R}^n$.]

SOLUTION: Let $p \in M$, and let $U \subset M$ be an open neighborhood of $p$ which is homeomorphic to an open subset in $\mathbb{R}^n$. Then we have $H_*(M, M - \{p\}) \cong H_*(U, U - \{p\})$ by the localization principle, so that topological invariance and excision imply the latter group is homeomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{0}\})$, which is infinite cyclic in dimension $n$ and zero otherwise.■

($b$) Prove that if $M$ is a topological $m$-manifold and $N$ is a topological $n$-manifold. Then $M$ is homeomorphic to $N$ only if $m = n$.

SOLUTION: If $K$ is a $k$-manifold then the first part shows that the local homology groups at every point are infinite cyclic in dimension $k$ and zero elsewhere. If $K$ is homeomorphic to both an $n$-manifold and and $m$-manifold, this implies that there is exactly one nonzero local homology group, which is in dimension $n$ if $K$ is an $n$-manifold and in dimension $m$ if $K$ is an $m$-manifold. The only way these can both happen is if $m = n$.■

($c$) Suppose that $f : M \to N$ is a 1–1 continuous mapping of topological $n$-manifolds. Prove that $f$ is an open mapping. [*Hint:* Why does it suffice to prove that each $p \in M$ has an open neighborhood $U_p$ such that $f|U_p$ is 1–1? Each point $f(p)$ has an open neighborhood $V_p$ which is homeomorphic to an open subset of $\mathbb{R}^n$. Why is there a neighborhood of $p$ which is also homeomorphic to an open subset of $\mathbb{R}^n$ and is mapped into $V_p$ by $f$?]

SOLUTION: To see the first sentence in the hint, note that if $W \subset M$ is an open subset and the condition in the hint is true, then the restrictions of $f$ to the open subsets $W \cap U_p$ are all open. Since

$$f[W] \quad = \quad \bigcup_p f[W \cap U_p]$$

and the maps $f|W \cap U_p$ are all open, this implies that $f[W]$ is also open.

To construct $U_p$ proceed as suggested in the hint. By continuity there is some open neighborhood $N_p$ of $p$ such that $f[N_p] \subset V_p$, and in $N_p$ there is some open neighborhood $U_p$ of $p$ which is homeomorphic to an open subset of $\mathbb{R}^n$. By construction the restriction of $f$ to $U_p$ is 1–1, and since $f$ maps $U_p$ into $V_p$ one can use invariance of domain to deduce that $f[U_p] \subset V_p$ is open.■

($d$) Prove that $S^n$ is not homeomorphic to a subset of $\mathbb{R}^n$. — In nonmathematical terms, this means that one cannot continuously flatten out a deflated beach ball on a table without some overlapping of points.■

SOLUTION: Suppose that such a homeomorphism $h$ existed. Then by the preceding part of the problem we know that the image of $h$ is an open subset of $\mathbb{R}^n$. However, since $S^n$ is compact, this image is also a closed subset, so that by connectedness $h[S^n]$ must be all of $\mathbb{R}^n$ and hence $S^n$ and $\mathbb{R}^n$ are homeomorphic ($h$ is a homeomorphism onto its image). This is clearly falso, so no homeomorphism like $h$ can exist.■

**3.** Let $(A_*, d_*)$ be a chain complex (say over the category of abelian groups). A **multiplicative structure** on $(A_*, d_*)$ is a family of bilinear mappings

$$\varphi_{p,q} : A_p \times A_q \longrightarrow A_{p+q}$$

which is a homomorphism in each variable with the other held constant and satisfies the following version of the Leibniz rule:

$$d\varphi(a_p, a_q) \quad = \quad \varphi(d(a_p), a_q) \; + \; (-1)^p \varphi(a_p, d(a_q))$$

Usually it is convenient to denote $\varphi(x, y)$ by notation such as $x * y$.

($a$) Prove that $\varphi$ induces a family of bilinear mappings

$$\varphi_* : H_p(A) \times H_q(A) \longrightarrow H_{p+q}(A)$$

such that if $u$ and $v$ are represented by cycles $x$ and $y$, then $x*y$ is a cycle and $u*v$ is represented by $x * y$. The proof should include justifications of the following assertions (this list is not necessarily exhaustive):

(1) If $x$ and $y$ are cycles then so is $x * y$.

(2) If $x = dw$ and $y$ is a cycle then $x * y$ is a boundary. Likewise, if $x$ is a cycle and $y = dv$ then $x * y$ is a boundary.

SOLUTION: The first major steps are to prove that the proposed definition

$$\varphi_*([u], [v]) \quad = \quad [\varphi(u, v)]$$

has property (1) and does not depend upon the choices of $u$ and $v$. To see that $du = 0$ and $dv = 0$ imply $d\varphi(u, v) = 0$, expand the left hand side using the Leibniz-like identity:

$$d(u * v) \quad = \quad (du) * v \; + \; (-1)^p u * (dv)$$

This follows quickly because bilinearity implies $a * 0$ and $0 * b$ are both zero, so $du = 0$ and $dv = 0$ imply that the right hand side of the displayed equation becomes $0 * v \pm u * 0 = 0$. Next, we need to show that if $u$ and $u'$ represent the same homology class and similarly for $v$ and $v'$, then $[u * v] = [u' * v']$. The assumptions imply that $u - u' = dw$ and $v - v' = dx$ for suitable $w$ and $x$. Therefore we have $[u' * v] = [(u + dw) * v] = [(u * v) + (dw * v)]$ by blinearity, and since passage to equivalence classes is additive the right hand side of these equations becomes $[u * v] + [(dw) * v]$; if we know that $(dw) * v$ is a boundary, then the sum will reduce to $[u * v]$ and we will have shown that this class only depends upon the homology class of the cycle $u$. But now we can use the Leibniz-like identity to conclude that

$$d(w * v) \quad = \quad (dw) * v \ \pm \ w * (dv) \quad = \quad (dw) * v \ \pm \ w * 0 \quad = \quad (dw) * v$$

which means that $(dw) * v$ is a boundary.

Similarly, if we can show that $du = 0$ implies $u * (dx)$ is a boundary, it will follow that $[u * v]$ also depends only upon the homology class of $v$. But now we have

$$d(u * x) \quad = \quad (du) * x * \ \pm u * (dx) \quad = \quad 0 * x \ \pm \ u * (dx) \quad = \quad \pm u * (dx)$$

so that $u * (dx) = d(\pm u * x)$ and this completes the verification that $\varphi_*$ is well defined.

To prove bilinearity, recall that the addition in homology satisfies $[y_1] + [y_2] = [y_1 + y_2]$, and since $[(u_1 + u_2) * v] = [(u_1 * v) + (u_2 * v)]$ by bilinearity the right hand side is just $[u_1 * v] + [u_2 * v]$, proving linearity in the first variable. Similarly we have

$$[u, (v_1 + v_2)] \quad = \quad [(u * v_1) + (u * v_2)] \quad = \quad [u * v_1] \ + \ [u, v_2]$$

yielding linearity in the second variable as well.■

($c$) Prove that the multiplicative structure in homology satisfies the associative law $(u*v)*w = u * (v * w)$ if the multiplicative structure on the chain complex level has this property.

SOLUTION: We want to prove that $([u] * [v]) * [w]) = [u] * ([v] * [w])$ if the multiplicative structure is associative on the chain level, and this is a consequence of the following sequence of equations:

$$([u]*[v])*[w] \quad = \quad [u*v]*[w] \quad = \quad [(u*v)*w] \quad = \quad [u*(v*w)] \quad = \quad [u]*[v*w] \quad = \quad [u]*([v]*[w])$$

where the crucial step is given by the associativity relation $[u * (v * w)] \quad = \quad [(u * v) * w]$.■

($d$) A two-sided unit for a multiplicative structure is a class $e \in A_0$ such that $de = 0$ and $e * a = a = a * e$ for all $a$. Prove that the homology class of $e$ is a two-sided unit for the multiplicative structure in homology and that this class is nontrivial if $H_q(A) \neq 0$ for some $q \neq 0$.

SOLUTION: Since $e * a = a = a * e$ for all $a$, we have

$$[e] * [a] \quad = \quad [e * a] \quad = \quad [a] \quad = \quad [a * e] \quad = \quad [a] * [e]$$

which shows taht $[e] \in H_0(A)$ is a two sided multiplicative unit. Bilinearity implies that $[u]*[v] = 0$ if either $u$ or $v$ is trivial in homology (see the preceding parts of this exercises), so if $[u] * [v]$ is nonzero then both $[u]$ and $[v]$ must be nonzero. Choose $a$ such that $[a] \neq 0$, and notice that we then have $0 \neq [a] = [e] * [a]$, which means that $[e]$ must define a nonzero homology class.■

**COMMENTS ON THE LAST EXERCISE.**

**1.** (For those familiar with differential forms.) The sign factor in the Leibniz-like identity might seem arbitrary, but there are counterparts in related constructions. For example, if one considers differential forms over open subsets of some $\mathbb{R}^n$, then the exterior derivative of the wedge product of a $p$-form and $q$-form satisfies the rule

$$d(\omega \wedge \theta) \quad = \quad (d\omega) \wedge \theta \ + \ (-1)^p \, \omega \wedge (d\theta) \ .$$

**2.** The corresponding commutative law takes the form

$$a_q * a_p \quad = \quad (-1)^{pq} a_p * a_q$$

but things are more complicated because there are many structures which are not exactly commutative on the chain level but are so in homology. In fact, such commutativity failures on the chain level reflect a fundamental difficulty in algebraic topology which can be managed effectively but leads to monumental complications in the subject.

**3.** Chaper 3 of Hatcher contains a far more extensive study of multiplicative structures in homology theory.