CHAPTER VII

HYPERQUADRICS

Conic sections have played an important role in projective geometry almost since the beginning of the subject. In this chapter we shall begin by defining suitable projective versions of conics in the plane, quadrics in 3-space, and more generally hyperquadrics in n-space. We shall also discuss tangents to such figures from several different viewpoints, prove a geometric classification for conics similar to familiar classifications for ordinary conics and quadrics in $\mathbb{R}^2$ and $\mathbb{R}^3$, and we shall derive an enhanced duality principle for projective spaces and hyperquadrics. Finally, we shall use a mixture of synthetic and analytic methods to prove a famous classical theorem due to B. Pascal (1623–1662)[1] on hexagons inscribed in plane conics, a dual theorem due to C. Brianchon (1783–1864),[2] and several other closely related results.

1. Definitions

> The three familiar curves which we call the "conic sections" have a long history ... It seems that they will always hold a place in the curriculum. The beginner in analytic geometry will take up these curves after he has studied the circle. Whoever looks at a circle will continue to see an ellipse, unless his eye is on the axis of the curve. The earth will continue to follow a nearly elliptical orbit around the sun, projectiles will approximate parabolic orbits, [and] a shaded light will illuminate a hyperbolic arch. — **J. L. Coolidge (1873–1954)**

In classical Greek geometry, conic sections were first described synthetically as intersections of a plane and a cone. On the other hand, today such curves are usually viewed as sets of points (x, y) in the Cartesian plane which satisfy a nontrivial quadratic equation of the form

$$Ax^2 \; + \; 2Bxy \; + \; Cy^2 \; + \; 2D \; + \; 2E \; + \; F \; = \; 0$$

where at least one of $A,\ B,\ C$ is nonzero. In these notes we shall generally think of conics and quadrics in such terms. Here are some online references which relate the classical and modern approaches to these objects. The first contains some historical remarks, the second is a fairly detailed treatment which shows the equivalence of the classical and modern definitions only using material from elementary geometry, and the third contains a different proof that the definitions are equivalent using standard results from trigonometry.

`http://xahlee.org/SpecialPlaneCurves_dir/ConicSections_dir/conicSections.html`

`http://mathdl.maa.org/convergence/1/?pa=content&sa=viewDocument&nodeId=196&bodyId=60`

[1]Incidentally, he proved this result when he was 16 years old.

[2]This result was originally discovered without using duality.

`http://math.ucr.edu/∼res/math153/history04Y.pdf`

The corresponding notion of *quadric surface* in $\mathbb{R}^3$ is generally defined to be the set of zeros of a nontrivial quadratic polynomial $p(x, y, z)$ in three variables (nontriviality means that at least one term of degree two has a nonzero coefficient). One can similarly define a *hyperquadric* in $\mathbb{R}^n$ to be the set of zeros of a nonzero quadratic polynomial $p(x_1, \cdots, x_n)$. Such an equation has the form

$$\sum_{i,j} a_{i,j} x_i x_j \ + \ 2 \cdot \sum_k b_k x_k \ + \ c \ = \ 0$$

where at least one of the coefficients $a_{i,j} \ = \ 0$.

One obvious question about our definitions is to give a concise but useful description of all the different types of conics, quadrics or hyperquadrics that exist in $\mathbb{R}^n$. Using linear algebra, in each dimension it is possible to separate or classify such objects into **finitely many** types such that

> *if Σ_1 and Σ_2 are hyperquadrics that are* affinely equivalent *(so that there is an affine transformation T of $\mathbb{R}^n$ such that $T[\Sigma_1] \ = \ \Sigma_2$, then Σ_1 and Σ_2 have the same type.* — In fact, *one can choose the affine transformation to have the form $T_1 \circ T_0$, where T_0 is a linear transformation and T_1 is given by a* diagonalizable *invertible linear transformation*; in other words, *there are nonzero scalars d_i such that for each i we have $T_1(\mathbf{e}_i) \ = \ d_i \mathbf{e}_i$, where $\mathbf{e}_i$ is the i^{th} standard unit vector in $\mathbb{R}^n$.*

For $n = 2$ and 3, the details of this classification are described explicitly in Section V.2 of the following online document:

`http://math.ucr.edu/∼res/math132/linalgnotes.pdf`

The case of conics in $\mathbb{R}^2$ is summarized in the table on page 82 of this document, and the case of quadrics in $\mathbb{R}^3$ is summarized in the table on page 83 of the same document. In particular, there are fewer than 10 different types of possible nonempty figures in $\mathbb{R}^2$ (including degenerate cases of sets with one point or no points) and fewer than 20 different types of possible nonempty figures in $\mathbb{R}^3$ (also including an assortment degenerate cases). Later in this chapter we shall describe the analogous classification for $\mathbb{R}^n$ (with $n \geq 3$ arbitrary) in one of the exercises.

Projective extensions of hyperquadrics

We are now faced with an obvious question:

> *How does one define a hyperquadric in projective space?*

Let us consider the analogous situation in degree one. The sets of solutions to nontrivial linear equations $p(x_1, \cdots, x_n) = 0$ are merely hyperplanes. If $p(x_1, \cdots, x_n)s \ = \ \sum_i \ a_i x_i \ + \ b$, then this hyperplane is just the set of ordinary points in $\mathbb{RP}^n$ whose homogeneous coordinates satisfy the *homogeneous* linear equation

$$\sum_{i=1}^{n} \ a_i x_i \ + \ b x_{n+1} \ = \ 0 \ .$$

This suggests the following: Consider the quadratic polynomial

$$p(x_1, \ \cdots \ , x_n) \ = \ \sum_{i,j} \ a_{i,j} x_i x_j \ + \ 2 \cdot \sum_k \ b_k x_k \ + \ c$$

and turn it into a homogeneous quadratic polynomial by multiplying each degree 1 monomial in the summation by x_{n+1} and multiplying the constant term by x_{n+1}^2. We then obtain the modified quadratic polynomial

$$\overline{p}(x_1, \cdots, x_n) \;=\; \sum_{i,j} a_{i,j} x_i x_j \;+\; 2 \cdot \sum_k b_k x_k x_{n+1} \;+\; c x_{n+1}^2$$

which is homogeneous and has the following compatibility properties:

THEOREM VII.1. *(i) If X is a point in $\mathbb{RP}^n$ and ξ and ξ' are homogeneous coordinates for X, then $\overline{p}(\xi) = 0$ if and only if $\overline{p}(\xi') = 0$.*

(ii) The set of zeros for p is equal to the set of ordinary points in $\mathbb{RP}^n$ whose homogeneous coordinates are zeros of $\overline{p}$.

Proof. We shall proof the two parts separately.

PROOF OF (i). Observe that $\overline{p}(k\xi) \;=\; k^2 \cdot \overline{p}(\xi)$ by direct computation. Therefore $\xi' = k\xi$ for some $k \neq 0$ implies that $\overline{p}(\xi') \;=\; 0$ if and only if $\overline{p}(\xi) \;=\; 0$.$\square$

PROOF OF (ii). If $\mathbf{x} \in \mathbb{R}^{n,1}$, then the transpose of $(x_1, \cdots, x_n, 1)$ is a set of homogeneous coordinates for $\mathrm{J}(\mathbf{x}) \in \mathbb{RP}^n$, and it is elementary to check that the solutions to the equation $p = 0$ contained in the intersection of the set of ordinary points and the points in $\mathbb{RP}^n$ whose homogeneous coordinates are solutions to the equation $\overline{p} = 0$ (in particular, we have $p(x_1, \cdots, x_n) \;=\; \overline{p}(x_1, \cdots, x_n, 1)$). Conversely, if $\overline{p}(x_1, \cdots, x_n, x_{n+1}) = 0$ where $x_{n+1} \neq 0$, then we also have

$$0 \;=\; \frac{1}{x_{n+1}^2} \cdot \overline{p}(x_1, \cdots, x_n, x_{n+1}) \;=\; \overline{p}\left(\frac{x_1}{x_{n+1}}, \cdots, \frac{x_n}{x_{n+1}}, 1\right) \;=\; p(x_1, \cdots, x_n)$$

and hence the solutions to $\overline{p} = 0$ in the image of J are all ordinary points which are solutions to $p = 0$.$\blacksquare$

All of the preceding discussion makes at least formal sense over an arbitrary field $\mathbb{F}$; of course, the mathematical value of the quadrics considered depends strongly upon the solvability of quadratic equations within the given field.[3] Define a *hyperquadric* Σ in $\mathbb{FP}^n$ to be the set of zeros of a homogeneous quadratic equation:

$$\sum_{i,j=1}^{n+1} a_{i,j} x_i x_j \;=\; 0$$

In the study of hyperquadrics we generally assume that $1 + 1 \neq 0$ in $\mathbb{F}$. This condition allows us to choose the n^2 coefficients $a_{i,j}$ so that $a_{i,j} = a_{j,i}$; for if we are given an arbitrary homogeneous quadratic equation as above and set $b_{i,j} \;=\; \frac{1}{2}(a_{j,i} + a_{i,j})$, then it is easy to see that

$$\sum_{i,j=1}^{n+1} a_{i,j} x_i x_j \;=\; 0 \text{ if and only if } \sum_{i,j=1}^{n+1} b_{i,j} x_i x_j \;=\; 0$$

because we have

$$\sum_{i,j=1}^{n+1} b_{i,j} x_i x_j \;=\; \tfrac{1}{2} \left(\sum_{i,j=1}^{n+1} a_{i,j} x_i x_j \;+\; \sum_{i,j=1}^{n+1} a_{j,i} x_i x_j \right) .$$

[3]All fields in this chapter are assumed to have commutative multiplications.

For these reasons, *we shall henceforth assume* $1 + 1 \neq 0$ *in* $\mathbb{F}$ *and* $a_{i,j} = a_{j,i}$ *for all* i *and* j.

It is natural to view the coefficients $a_{i,j}$ as the entries of a *symmetric* $(n+1) \times (n+1)$ matrix A. If we do so and Σ is the hyperquadric in $\mathbb{FP}^n$ defined by the equation $\sum_{i,j} a_{i,j} x_i x_j = 0$, then we may rewrite the defining equation for Σ as follows: A point X lies on Σ if and only if for some (equivalently, for all) homogeneous coordinates ξ representing X we have

$$ {}^{\mathbf{T}}\xi A \xi \;=\; 0 \; . $$

If we have an affine quadric in $\mathbb{F}^n$ defined by a polynomial p as above, then an $(n+1) \times (n+1)$ matrix defining its projective extension is given in block form by

$$ \begin{pmatrix} A & {}^{\mathbf{T}}\mathbf{b} \\ \mathbf{b} & c \end{pmatrix} $$

where the symmetric matrix $A = (\, a_{i,j} \,)$ gives the second degree terms of p, the row vector $2 \cdot \mathbf{b}$ gives the first degree terms b_i (note the coefficient!), and c gives the constant term.

Hypersurfaces of higher degree

The reader should be able to define projective hypercubics, hyperquartics, *etc.*, as well as the projective hyper—ic associated to an affine hyper—ic. Subsets of these types are generally called **projctive algebraic varieties**; they have been studied extensively over the past 300 years and have many interesting and important properties. The mathematical study of such objects has remained an important topic in mathematics ever since the development of projective geometry during the 19^{th} century, but it very quickly gets into issues far beyond the scope of these notes. In particular, the theory involves a very substantial amount of input from multivariable calculus and the usual approaches also require considerably more sophisticated algebraic machinery than we introduce in these notes. The rudiments of the theory appear in Sections V.4–V.6 of the book by Bumcrot, and a more complete treatment at an advanced undergraduate level is given in Seidenberg, *Elements of the Theory of Algebraic Curves*, as well as numerous other introductory books on **algebraic geometry.**

Projective algebraic varieties also turn out to have important applications in various directions, including issues in theoretical physics, the theory of encryption, and even the proof of Fermat's Last Theorem during the 1990s which was mainly due to Andrew Wiles (the word "mainly" is included because the first complete proof required some joint work of Wiles with R. Taylor, and Wiles' work starts with some important earlier results by others). A reader who wishes to learn more about some of these matters may do so by going to the final part of Section IV.5 in the online document

`http//:math.ucr.edu∼res/math133/coursenotes4b.pdf`

and checking the traditional and electronic references cited there.

EXERCISES

1. Consider the conics in $\mathbb{R}^2$ defined by the following equations:

(i) The circle defined by $x^2 + y^2 - 1 = 0$.

(ii) The hyperbola defined by $xy - 1 = 2$.

(iii) The parabola defined by $y - x^2 = 0$.

Show that the associated projective conics have 0, 1 and 2 points at infinity respectively, and give homogeneous coordinates for these points.

2. Find which points (if any) at infinity belong to the projective conics associated to the conics in $\mathbb{R}^2$ defined by the following equations.

(i) $x^2 - 2y^2 - 2xy \;=\; 0$

(ii) $3x^2 + 4y^2 - 4x + 2 \;=\; 0$

(iii) $x^2 + y^2 - 4y \;=\; 4$

(iv) $x^2 - 4xy - 4y^2 - 2y \;=\; 4$

3. Find the points at infinity on the projective quadrics associated to the quadrics in $\mathbb{R}^3$ defined by the following equations.

(i) $x^2 + y^2 - z^2 \;=\; 1$

(ii) $x^2 + y^2 - z^2 - 6x - 8y \;=\; 0$

(iii) $x^2 + y^2 \;=\; 2z$

(iv) $x^2 - y^2 - z^2 \;=\; 1$

(v) $x^2 + y^2 \;=\; z$

(vi) $x^2 + y^2 \;=\; z^2$

4. For each of the following affine quadrics σ in $\mathbb{R}^3$, find a symmetric 4×4 matrix such that the projective extension $\mathbb{P}(\Sigma)$ of Σ is defined by the equation ${}^{\mathbf{T}}\xi A \xi \;=\; 0$.

(i) Σ is defined by the affine equation $4x^2 + 3y^2 - z^2 + 2x + y + 2z - 1 \;=\; 0$.

(ii) Σ is defined by the affine equation $3x^2 + y^2 + 2z^2 + 3x + 3y + 4z \;=\; 0$.

(iii) Σ is defined by the affine equation $2x^2 + 4z^2 - 4x - y - 24z + 36 \;=\; 0$.

(iv) Σ is defined by the affine equation $4x^2 + 9y^2 + 5z^2 - 4xy + 8yz + 12xz + 9z - 3 \;=\; 0$.

2. Tangents

Tangent lines to circles play an important role in classical Euclidean geometry, and their generalizations to other conics we also known to classical Greek mathematicians such as Archimedes (287 B. C. E. – 212 B. C. E.) and Apollonius of Perga (*c.* 262 B. C. E. – *c.* 190 B. C. E.). In modern mathematics they are generally defined using concepts and results from single variable or multivariable differential calculus. Of course, the latter is designed to work primarily in situations where the coordinates are real or complex numbers, and since we want to consider more general coordinates we need to develop an approach that is at least somewhat closer to the classical viewpoint.

In these notes we shall concentrate on the following two ways of viewing tangents to conics in $\mathbb{R}^2$ or quadrics in $\mathbb{R}^3$.

1. SYNTHETIC APPROACH. A line is tangent to a hyperquadric if and only if it lies wholly in the hyperquadric or has precisely one point of intersection with the hyperquadric.
2. ANALYTIC APPROACH. Let $X \in \Sigma \cap L$, where Σ is a hyperquadric and L is a line. Then L is tangent to Σ if and only if there is a differentiable curve $\gamma : (a;b) \to \mathbb{R}^n$ lying totally in Σ such that $\gamma(t_0) = \mathbf{x}$ for some $t_0 \in (a;b)$ and L is the line $\mathbf{x} \ + \ \mathbb{R} \cdot \gamma'(t_0)$.

For our purposes the first viewpoint will be more convenient; in Appendix E we shall show that the analytic approach is consistent with the synthetic viewpoint, at least in all the most important cases. Actually, the viewpoint of calculus is the better one for generalizing tangents to cubics, quartics, *etc.*, but a correct formulation is too complicated to be given in these notes.

We begin with a result on solutions to homogeneous quadratic equations in two variables:

THEOREM VII.2. *Suppose that $\mathbb{F}$ is a field in which $1+1 \neq 0$, and (x_1,y_1), (x_2,y_2), (x_3,y_3) are solutions to the homogeneous quadratic equation*

$$ax^2 \ + \ bxy \ + \ cy^2 \ = \ 0 \ .$$

Then either $a = b = c = 0$ or else one of (x_1,y_1), (x_2,y_2), (x_3,y_3) is a nonzero multiple of another.

Proof. If the hypothesis holds, then in matrix terminology we have

$$\begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 \\ x_2^2 & x_2y_2 & y_2^2 \\ x_3^2 & x_3y_3 & y_3^2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \ .$$

Suppose not all of $a, \ b, \ c$ are nonzero. Then the given 3×3 matrix is not invertible and hence has a zero determinant. But the determinant of such a matrix may be computed directly, and up to a sign factor it is equal to

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} \cdot \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

The vanishing of this determinant implies that one of the 2×2 determinants in the factorization must be zero, and the latter implies that the rows of the associated 2×2 matrix are proportional to each other.■

The preceding result has the following important geometric application:

THEOREM VII.3. *Let Σ be a hyperquadric in $\mathbb{RP}^n$, let $X \in \Sigma$, and let L be a line containing X. Then $\Sigma \cap L$ is either $\{X\}$, two points, or all of L.*

Proof. Let $Y \neq X$ where $Y \in L$, let ξ and η denote homogeneous coordinates for X and Y respectively, and suppose that Σ is defined by the equation

$$ {}^{\mathbf{T}}\omega A \omega \ = \ 0 $$

where A is a symmetric $(n+1) \times (n+1)$ matrix and ω represents $W \in \mathbb{RP}^n$.

If $Z \in L$ and is represented by the homogeneous coordinates ζ, then $\zeta \ = \ u\xi + v\eta$ for some $u, \ v \in \mathbb{F}$ that are not both zero. By construction, $Z \in \Sigma$ if and only if

$$ 0 \ = \ {}^{\mathbf{T}}\zeta A \zeta \ = \ {}^{\mathbf{T}}(u\xi + v\eta) A (u\xi + v\eta) \ = $$

$$ u^2\,{}^{\mathbf{T}}\xi A \xi \ + \ 2uv\,{}^{\mathbf{T}}\eta A \xi \ + \ v^2\,{}^{\mathbf{T}}\eta A \eta \ = \ u^2 p \ + \ 2uvq \ + \ v^2 q $$

for suitable constants $p, \ q, \ r$. We claim that $\Sigma \cap L$ has at least three points if and only if $L \subset \Sigma$. The "only if" implication is trivial, so we shall focus on the "if" direction. — Suppose that $Z_1, \ Z_2, \ Z_3$ are points on $\Sigma \cap L$, and take homogeneous coordinates $\zeta_i \ = \ u_i\xi + v_i\eta$ for Z_i. By Theorem 2, either $p = q = r = 0$ (in which case $L \subset \Sigma$) or else one of the pairs (u_i, v_i) is proportional to the other, say $(u_j, v_j) = m(u_k, v_k)$ for some $m \neq 0$. In this case we have that $Z_j = Z_k$ and hence $Z_1, \ Z_2, \ Z_3$ are not distinct.■

Definition. Let Σ be a hyperquadric, let $X \in \Sigma$, and let L be a line containing X. We shall say that L is a *tangent line* to Σ at X if either $\Sigma \cap L - \{X\}$ or $L \subset \Sigma$. In the remaining case where $\Sigma \cap L$ consists of two points, we shall say that L is a *secant line through* X. The *tangent space* to Σ at X is equal to the union of all tangent lines to Σ at X.

Singular and nonsingular points

If we consider the conic in $\mathbb{R}^2$ defined by the eqution $x^2 - y^2 = 0$ we see that the structure of the conic at the origin is different than at other points, for the conic is given by a pair of lines which intersect at the origin. Some words which may be used to describe this difference are exceptional, special or *singular*. A concise but informative overview of singular points for plane curves appears in the following online reference:

`http://mathworld.wolfram.com/SingularPoint.html`

There are corresponding theories of singularities for surfaces in $\mathbb{R}^3$, and more generally for hypersurfaces in $\mathbb{R}^n$. Not surprisingly, if one is only interested in hyperquadrics as in these notes, then everything simplifies considerably. We shall explain the relationship between the theory of singular and nonsingular points for hyperquadrics and the general case in Appendix E.

We have given a purely synthetic definition of the tangent space to a hyperquadric $\Sigma \subset \mathbb{FP}^n$ at a point $X \in \Sigma$. The first step is to give an algebraic description of the tangent space in terms of homogeneous coordinates.

THEOREM VII.4. *Let $\mathbb{F}$ and $\Sigma \subset \mathbb{FP}^n$ be as above, and let $X \in \Sigma$. Then the tangent space to Σ at X is either a hyperplane in $\mathbb{FP}^n$ or all of $\mathbb{FP}^n$. In the former case, X is said to be a* nonsingular *point, and in the latter case X is said to be a* singular point. *Furthermore, if Σ is defined by the symmetric matrix A and ξ is a set of homogeneous coordinates for X, then in the nonsingular case ${}^{\mathbf{T}}\xi A$ is a (nonzero) set of homogeneous coordinates for the tangent hyperplane, but in the singular case we have ${}^{\mathbf{T}}\xi A \ = \ \mathbf{0}$.*

EXAMPLES. Suppose we consider the projectivizations of the circle $x^2 + y^2 = 1$, the hyperbola $x^2 - y^2 = 1$, the parabola $y = x^2$, and the pair of intersecting lines $x^2 = y^2$. Then the corresponding projective conics are defined by the following homogeneous quadratic equations:

$$x_1^2 + x_2^2 - x_3^2 \ = \ 0, \qquad\qquad x_1^2 - x_2^2 - x_3^2 \ = \ 0$$

$$x_1^2 - x_2 x_3 \ = \ 0, \qquad\qquad x_1^2 - x_2^2 \ = \ 0$$

In the first three cases the associated 3×3 symmetric matrix A is invertible, and hence ${}^{\mathbf{T}}\xi A \neq \mathbf{0}$ for all nonzero ξ, so that every point of these projective conics will be a nonsingular point. — On the other hand, in the fourth example, the symmetric matrix A is not invertible, and in fact its kernel (either on the left or right side!) consists of all vectors whose first and second coordinates are equal to zero. This implies that all points on the conic except $\mathrm{J}(\mathbf{0})$ are nonsingular but $\mathrm{J}(\mathbf{0})$ is singular. These examples are all consistent with our intuition that the first three curves behave regularly (or are nonsingular) at all points and the fourth curve behaves regularly at all points except the origin.

Proof. In the proof of the preceding theorem, we noted that if $Y \in \mathbb{FP}^n$ with homogeneous coordinates η and $Z \in XY$ has homogeneous coordinates $\zeta \ = \ u\xi + v\eta$, then $Z \in \Sigma$ if and only if

$$u^2 \left({}^{\mathbf{T}}\xi A \xi\right) \ + \ 2uv \left({}^{\mathbf{T}}\xi A \eta\right) \ + \ v^2 \left({}^{\mathbf{T}}\eta A \eta\right) \ = \ 0$$

and the number of points on $XY \cap \Sigma$ depends upon the equivalence classes of solutions to this equation, which we shall call the INTERSECTION EQUATION.

CLAIM: The line XY is tangent to Σ if and only if ${}^{\mathbf{T}}\xi A \eta \ = \ {}^{\mathbf{T}}\eta A \xi \ = \ 0$.

Suppose first that XY is tangent to Σ. If XY is contained in Σ, then we have

$${}^{\mathbf{T}}\xi A \xi \ = \ {}^{\mathbf{T}}\eta A \eta \ = \ {}^{\mathbf{T}}(\xi + \eta) A (\xi + \eta) \ = \ 0$$

and elementary manipulations of these equations show that $2 \cdot {}^{\mathbf{T}}\eta A \xi \ = \ 0$. On the other hand, if $XY \cap \Sigma \ = \ \{X\}$, then ${}^{\mathbf{T}}\eta A \eta \ = \ 0$ and the only solutions to the Intersection Equation in the first paragraph of the proof are pairs (u, v) which are nonzero scalar multiples of $(1, 0)$. Therefore, the Intersection Equation evaluated at $(1, t)$ is equal to zero if and only if $t = 0$. However, it is easy to check that the ordered pair

$$\left(1, \ -\frac{{}^{\mathbf{T}}\xi A \eta}{{}^{\mathbf{T}}\eta A \eta}\right)$$

solves the Intersection Equation because

$${}^{\mathbf{T}}\xi A \xi \ = \ 0$$

and therefore we must have ${}^{\mathbf{T}}\xi A \eta \ = \ {}^{\mathbf{T}}\eta A \xi \ = \ 0.\square$

Conversely, suppose that ${}^{\mathbf{T}}\xi A \eta \ = \ {}^{\mathbf{T}}\eta A \xi \ = \ 0$. Since ${}^{\mathbf{T}}\xi A \xi \ = \ 0$, the Intersection Equation reduces to

$$v^2 \left({}^{\mathbf{T}}\eta A \eta\right) \ = \ 0 \ .$$

This equation means that either ${}^{\mathbf{T}}\eta A\eta \;=\; 0$, in which case we have $L \subset \Sigma$, or else $v = 0$, in which case every solution (u, v) of the Intersection equation is proportional to the known solution $(1, 0)$, so that $\Sigma \cap XY \;=\; \{X\}$.

To conclude the proof, we have shown that the tangent space at X is the set of all points Y such that ${}^{\mathbf{T}}\xi A\eta \;=\; 0$. If ${}^{\mathbf{T}}\xi A \;=\; \mathbf{0}$, this is all of $\mathbb{FP}^n$, and if ${}^{\mathbf{T}}\xi A \;\neq\; \mathbf{0}$, this is the hyperplane with homogeneous coordinates ${}^{\mathbf{T}}\xi A$.■

We shall say that a hyperquadric Σ is **nonsingular** if for each $X \in \Sigma$ the tangent space at X is a hyperplane (algebraically, this means that if ξ represents X then ${}^{\mathbf{T}}\xi A \neq \mathbf{0}$.

THEOREM VII.5. *If Σ is a hyperquadric defined by the symmetric matrix A, then Σ is nonsingular if and only if A is invertible.*

Proof. Suppose first that A is invertible. Then $\xi \neq \mathbf{0}$ implies that ${}^{\mathbf{T}}\xi A$ is nonzero, and by the preceding result it follows that the tangent space at every point must be a hyperplane.□

Conversely, suppose that A is not invertible. Then there is some $\xi \neq \mathbf{0}$ such that ${}^{\mathbf{T}}\xi A \;=\; \mathbf{0}$, and if ξ represents X it follows that $X \in \Sigma$ and X is a singular point of Σ.■

By definition, each symmetric matrix A determines a hyperquadric Σ_A. This is not a 1–1 correspondence, for if c is a nonzero scalar then clearly $\Sigma_A \;=\; \Sigma_{cA}$. We shall now use the notion of tangent hyperplane to show that, in many cases, this is the only condition under which two matrices can define the same hyperquadric. Further discussion of this question is given in Section 2 of Appendix E.

THEOREM VII.6. *Let A and B be symmetric $(n+1) \times (n+1)$ matrices over the field $\mathbb{F}$ in which $1 + 1 \neq 0$, and suppose they define the same nonempty hyperquadric in $\mathbb{FP}^n$. Assume that Σ has at least one nonsingular point. Then B is a scalar multiple of A.*

Proof. We are given that Σ has a nonsingular point X; let ξ be a set of homogeneous coordinates for X. Then both ${}^{\mathbf{T}}\xi A$ and ${}^{\mathbf{T}}\xi B$ define the same hyperplane and hence ${}^{\mathbf{T}}\xi A \;=\; k \cdot {}^{\mathbf{T}}\xi B$ for some nonzero scalar k.

Suppose now that Y does not lie on this tangent hyperplane, and let η be a set of homogeneous coordinates for Y. Then the line XY meets Σ in a second point which has homogeneous coordinates of the form $u\xi + \eta$ for some $u \in \mathbb{F}$. This scalar satisfies the following equations:

$$2u\,{}^{\mathbf{T}}\xi A\eta \;+\; {}^{\mathbf{T}}\eta A\eta \;=\; 0, \qquad\qquad 2u\,{}^{\mathbf{T}}\xi B\eta \;+\; {}^{\mathbf{T}}\eta B\eta \;=\; 0$$

Since ${}^{\mathbf{T}}\xi A \;=\; k \cdot {}^{\mathbf{T}}\xi B$ the equations above imply that

$${}^{\mathbf{T}}\eta A\eta \;=\; k \cdot {}^{\mathbf{T}}\eta B\eta$$

for all Y whose homogeneous coordinates satisfy ${}^{\mathbf{T}}\xi A\eta \neq 0$ (*i.e.*, all vectors in $\mathbb{F}^{n+1,1}$ except those in the n-dimensional subspace defined by the tangent hyperplane to Σ and X).

To prove that ${}^{\mathbf{T}}\eta A\eta \;=\; k \cdot {}^{\mathbf{T}}\eta B\eta$ if Y lies in the tangent hyperplane at X, let Z be a point which is not on the tangent hyperplane. Then

$${}^{\mathbf{T}}\omega A\omega \;=\; k \cdot {}^{\mathbf{T}}\omega B\omega$$

for $\omega = \zeta,\ \eta+\zeta,\ \eta-\zeta$. Let $C = A$ or B, and write $\Psi_C(\gamma,\delta) = {}^{\mathbf{T}}\gamma C\delta$. We then have the following:

$$\psi_C(\eta,\zeta) \;=\; \tfrac{1}{4}\Psi_C(\eta+\zeta,\eta+\zeta) \;-\; \tfrac{1}{4}\Psi_C(\eta-\zeta,\eta-\zeta)$$

$$\Psi_C(\eta,\eta) \;=\; \Psi_C\big(\,(\eta+\zeta)-\zeta,\,(\eta+\zeta)-\zeta\,\big)$$

By the first of these and the preceding paragraph, we have $\Psi_A(\eta,\zeta) = k\cdot\Psi_B(\eta,\zeta)$. Using this, the second equation above and the preceding paragraph, we see that $\Psi_A(\eta,\eta) = k\cdot\Psi_B(\eta,\eta)$ if η represents a point Y in the tangent hyperplane to Σ at X. Applying this and the first displayed equation to arbitrary nonzero vectors $\eta,\ \zeta \in \mathbb{F}^{n+1,1}$, we see that $\Psi_A(\eta,\zeta) = k\cdot\Psi_B(\eta,\zeta)$. Since $c_{i,j}$ is the value of $\Psi_C(\mathbf{e}_i,\mathbf{e}_j)$ if $\mathbf{e}_i$ and $\mathbf{e}_j$ are the standard unit vectors (the k^{th} coordinate of $\mathbf{e}_k$ is 1 and the rest are 0), we see that $a_{i,j} = k\cdot b_{i,j}$ for all i and j, and hence we see that $B = k\cdot A$.■

EXERCISES

In all these exercises $\mathbb{F}$ denotes a (commutative) field in which $1+1\neq 0$.

1. Find the singular points (if any) of the projective conics given in Exercise 3 of the previous section.

2. Find the equations of the tangent lines tot he following conics in $\mathbb{RP}^2$ at the indicated points:

(i) The conic defined by $x_1^2 + 2x_1x_2 + 4x_1x_3 + 3x_2^2 - 12x_1x_3 + 2x_3^2 = 0$ at the points

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}1\\1\\3\end{pmatrix}\ .$$

(ii) The conic defined by $x_1^2 - 2x_1x_2 + 4x_2^2 - 4x_3^2 = 0$ at the points

$$\begin{pmatrix}2\\2\\-1\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}2\\0\\1\end{pmatrix}\ .$$

Definition. Let Σ be a hyperquadric in $\mathbb{FP}^n$ defined by the $(n+1)\times(n+1)$ matrix A such that Σ has at least one nonsingular point. Two points X and Y in $\mathbb{FP}^n$ are said to be *conjugate* with respect to Σ if they have homogeneous coordinates ξ and η respectively such that ${}^{\mathbf{T}}\xi A\eta = 0$. By Theorem 6, this definition does not depend upon any of the choices (including A). Moreover, a point is self-conjugate if and only if it lies on Σ.

3. In the setting above, assume that $X \notin \Sigma$ and Y is conjugate to P with respect to Σ. Suppose that $XY\cap\Sigma$ consists of two points, say A and B. Prove that $\mathsf{XR}(X,Y,A,B) = -1$.

Note. If Σ is nonsingular and nonempty (hence A is invertible by Theorem 5) and $X \in \mathbb{FP}^n$, then a hyperplan with homogeneous coordinates ${}^{\mathbf{T}}\xi A$ is called the *polar hyperplane* of X with respect to Σ. The map $\mathbf{P}$ sending X to its polar hyperplane is a collineation from $\mathbb{FP}^n$ to its dual $\left(\mathbb{FP}^n\right)^*$ is called a **polarity**, and it has the property that the composite

$$\mathbb{FP}^n \ \xrightarrow[\cong]{\mathbf{P}} \ (\mathbb{FP}^n)^* \ \xrightarrow[\cong]{\mathbf{P}^*} \ (\mathbb{FP}^n)^{**}$$

is the identity.

4. Let Σ be an affine hyperquadric in $\mathbb{F}^n$, where $n \geq 3$, and suppose that L is a line in $\mathbb{F}^n$ such that $L \subset \Sigma$. Denote the projective extension of Σ by Σ^*. Prove that the ideal point L_∞, and in fact the entire projective line

$$\mathrm{J}[L] \ \cup \ \{L_\infty\}$$

is contained in Σ^*. [*Hint:* The field $\mathbb{F}$ contains at least three elements. What does this imply about the number of points on L, and how does this lead to the desired conclusion?]

3. Bilinear forms

At this point it is convenient to discuss a topic in linear algebra which is generally not covered in first courses on the subject. For the time being, $\mathbb{F}$ will be a (commutative field with no assumption on whether or not $1+1=0$ or $1+1\neq 0$.

Definition. Let V be a vector space over $\mathbb{F}$. A *bilinear form* on $\mathbb{F}$ is a function

$$\Phi : V \times V \longrightarrow \mathbb{F}$$

with the following properties:

(Bi–1) $\Phi(\mathbf{v}+\mathbf{v}', \mathbf{w}) = \Phi(\mathbf{v}, \mathbf{w}) + \Phi(\mathbf{v}', \mathbf{w})$ for all $\mathbf{v},\ \mathbf{v}',\ \mathbf{w} \in V$.

(Bi–2) $\Phi(\mathbf{v}, \mathbf{w}+\mathbf{w}') = \Phi(\mathbf{v}, \mathbf{w}) + \Phi(\mathbf{v}, \mathbf{w}')$ for all $\mathbf{v},\ \mathbf{w},\ \mathbf{w}' \in V$.

(Bi–3) $\Phi(c\cdot\mathbf{v}, \mathbf{w}) = c\cdot\Phi(\mathbf{v}, \mathbf{w}) = \Phi(\mathbf{v}, c\cdot\mathbf{w})$ for all $\mathbf{v},\ ,\mathbf{w} \in V$ and $c \in \mathbb{F}$.

The reader will notice the similarities between the identities for Φ and the identities defining the dot product on $\mathbb{R}^n$. Both are scalar valued, distributive in both variables, and homogeneous (of degree 1) with respect to scalars. However, we are not assuming that Φ is commutative — in other words, we make no assumption about the difference between $\Phi(\mathbf{v}, \mathbf{w})$ and $\Phi(\mathbf{w}, \mathbf{v})$ — and we can have $\Phi(\mathbf{x}, \mathbf{x}) = 0$ even if $\mathbf{x}$ is nonzero.

EXAMPLES. **1.** Let $\mathbb{F}=\mathbb{R}$ and $V=\mathbb{R}^2$, and let $\Phi(\mathbf{x}, \mathbf{y}) = x_1y_2 - x_2y_1$, where by convention $\mathbf{a}\in\mathbb{R}^2$ can be written in coordinate form as (a_1, a_2). Then $\Phi(\mathbf{y}, \mathbf{x}) = -\Phi(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}$ and $\mathbf{y}$ and we also have $\Phi(\mathbf{z}, \mathbf{z}) = 0$ for all $\mathbf{z}\in\mathbb{R}^2$.

2. Let $\mathbb{F}$ and V be as above, and $\Phi(\mathbf{x}, \mathbf{y}) = x_1y_1 - x_2y_2$. In this case we have the commutativity identity $\Phi(\mathbf{y}, \mathbf{x}) = \Phi(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}$ and $\mathbf{y}$, but if $\mathbf{z}=(1,1)$, or any multiple of the latter, then $\Phi(\mathbf{z}, \mathbf{z}) = 0$.

3. Let A be an $n\times n$ matrix over $\mathbb{F}$, and let V be the vector space of all $n\times 1$ column matrices. Define a bilinear form Φ_A on V by the formula

$$\Phi_A(\mathbf{x}, \mathbf{y}) = {}^{\mathbf{T}}\mathbf{x}A\mathbf{y} .$$

Examples of this sort appeared frequently in the preceding section (see also Appendix E). Actually, the first two examples are special cases of this construction in which A is given as follows:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In fact, the following theorem shows that, in principle, the preceding construction gives **all** possible bilinear forms on finite-dimensional vector spaces.

THEOREM VII.7. *Let v be ann-dimensional vector space over $\mathbb{F}$, and let $\mathcal{A} = \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ be an ordered basis for V. If Φ is a bilinear form over $\mathbb{F}$, let $[\Phi]_{\mathcal{A}}$ be the $n\times n$ matrix whose (i,j) entry is equal to $\Phi(\mathbf{a}_i, \mathbf{a}_j)$. Then the map sending Φ to $[\Phi]_{\mathcal{A}}$ defines a $1-1$ correspondence between bilinear forms over V and $n\times n$ matrices over $\mathbb{F}$.*

The matrix $[\Phi]_{\mathcal{A}}$ is called *the matrix of* Φ *with respect to the ordered basis* $\mathcal{A}$.

Proof. *The mapping is* $1-1$. Suppose that we are given two bilinear forms Φ and Ψ such that $\Phi(\mathbf{a}_i, \mathbf{a}_j) = \Psi(\mathbf{a}_i, \mathbf{a}_j)$ for all i and j (this is the condition for $[\Phi]_{\mathcal{A}}$ and $[\Psi]_{\mathcal{A}}$ to be equal). If $\mathbf{v},\ \mathbf{w} \in V$, express these vectors as linear combinations of the basis vectors as follows:

$$\mathbf{v} \ = \ \sum_i \ x_i\mathbf{a}_i \qquad\qquad \mathbf{w} \ = \ \sum_j \ y_j\mathbf{b}_j$$

Then by **(Bi–1)** — **(Bi–3)** we have

$$\Phi(\mathbf{v},\mathbf{w}) \ = \ \sum_{i,j} \ x_iy_j\Phi(\mathbf{a}_i,\mathbf{a}_j) \ = \ \sum_{i,j} \ x_iy_j\Psi(\mathbf{a}_i,\mathbf{a}_j) \ = \ \Psi(\mathbf{v},\mathbf{w})$$

and since $\mathbf{v}$ and $\mathbf{w}$ are arbitrary we have $\Phi \ = \ \Psi$.□

The mapping is onto. If B is an $n \times n$ matrix and $\mathbf{v},\ \mathbf{w} \in V$ are as in the preceding paragraph, define

$$\mathbf{f}_{B,\mathcal{A}} \ = \ \sum_{i,j} \ x_iy_jb_{i,j} \ .$$

This is well-defined because the coefficients of $\mathbf{v}$ and $\mathbf{w}$ with respect to $\mathcal{A}$ are uniquely determined. The proof that $\mathbf{f}_{B,\mathcal{A}}$ satisfies **(Bi–1)** — **(Bi–3)** is a sequence of routine but slightly messy calculations, and it is left as an exercise. Given this, it follows immediately that B is equal to $[\mathbf{f}_{B\mathcal{A}}]_{\mathcal{A}}$.■

CHANGE OF BASIS FORMULA. Suppose we are given a bilinear form Φ on an n-dimensional vector space V over $\mathbb{F}$, and let $\mathcal{A}$ and $\mathcal{B}$ be ordered basis for V. In several contexts it is useful to understand the relationship between the matrices $[\Phi]_{\mathcal{A}}$ and $[\Phi]_{\mathcal{B}}$. The equation relating these matrices are given by the following result:

THEOREM VII.8. *Given two ordered bases* $\mathcal{A}$ *and* $\mathcal{B}$*, define a transition matrix by the form*

$$\mathbf{b}_j \ = \ \sum_i \ p_{i,j}\mathbf{a}_i \ .$$

If Φ *is a bilinear form on* V *as above, then we have*

$$[\Phi]_{\mathcal{B}} \ = \ {}^{\mathbf{T}}P[\Phi]_{\mathcal{A}}\,P \ .$$

Proof. We only need to calculate $\Phi(\mathbf{b}_i, \mathbf{b}_j)$; by the equations above, we have

$$\Phi(\mathbf{b}_i,\mathbf{b}_j) \ = \ \Phi\left(\sum_k \ p_{k,i}\mathbf{a}_k, \sum_m \ p_{m,j}\mathbf{a}_m\right) \ =$$

$$\sum_k \ \left(p_{k,i}\left(\sum_m \ p_{m,k}\,\Phi(\mathbf{a}_k,\mathbf{a}_m)\right)\right) \ .$$

However, the coefficient of $p_{k,i}$ is just the (k,j) entry of $[\Phi]_{\mathcal{A}}\,P$, and hence the entire summation is just the (i,j) entry of ${}^{\mathbf{T}}P[\Phi]_{\mathcal{A}}\,P$, as claimed.■

Definition. A bilinear form Φ is *symmetric* if $\Phi(\mathbf{x},\mathbf{y}) \ = \ \Phi(\mathbf{y},\mathbf{x})$ for all $\mathbf{x}$ and $\mathbf{y}$.

THEOREM VII.9. *Let* Φ *and* $\mathcal{A}$ *be as in Theorem* 7*. Then* Φ *is symmetric if and only if* $[\Phi]_{\mathcal{A}}$ *is a symmetric matrix.*

Proof. Suppose that Φ is symmetric. Then $\Phi(\mathbf{a}_i, \mathbf{a}_j) = \Phi(\mathbf{a}_j, \mathbf{a}_i)$ for all i and j, and this implies that $[\Phi]_{\mathcal{A}}$ is a symmetric matrix.

Conversely, if $[\Phi]_{\mathcal{A}}$ is symmetric and $\mathbf{v},\ \mathbf{w} \in V$ (the same notation as in Theorem 7), then by Theorem 7 we have

$$\Phi(\mathbf{v}, \mathbf{w}) \;=\; \sum_{i,j} \; ([\Phi]_{\mathcal{A}})_{i,j}\, x_i y_j \qquad\qquad \Phi(\mathbf{w}, \mathbf{v}) \;=\; \sum_{i,j} \; ([\Phi]_{\mathcal{A}})_{j,i}\, x_i y_j \;.$$

Since $[\Phi]_{\mathcal{A}}$ is symmetric, the two summations are equal, and therefore we must have

$$\Phi(\mathbf{y}, \mathbf{x}) \;=\; \Phi(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}$ and $\mathbf{y}$.$\blacksquare$

We have introduced all of the preceding algebraic machinery in order to prove the following result:

THEOREM VII.10. *Let $\mathbb{F}$ be a field in which $1 + 1 \neq 0$, and let A be a symmetric $n \times n$ matrix over $\mathbb{F}$. Then there is an invertible matrix P such that ${}^{\mathbf{T}}PAP$ is a diagonal matrix.*

This will be a consequence of the next result:

THEOREM VII.11. *Let Φ be a symmetric bilinear form on an n-dimensional vector space V over a field $\mathbb{F}$ for which $1 + 1 \neq 0$. Then there is an ordered basis $\mathbf{v}_1, \;\cdot\;, \mathbf{v}_n$ of V such that $\Phi(\mathbf{v}_i, \mathbf{v}_j) \;=\; 0$ if $i \neq j$ and $\Phi(\mathbf{v}_i, \mathbf{v}_i) \;=\; d_i$ for suitable scalars $d_i \in \mathbb{F}$.*

Proof that Theorem 11 implies Theorem 10. Define a bilinear form Φ_A as in Example 1 above. By construction $[\Phi_A]_{\mathcal{U}} \;=\; A$, where $\mathcal{U}$ is the ordered basis obtained of standard unit vectors. On the other hand, if $\mathcal{V}$ is the ordered basis obtained from Theorem 11, then $[\Phi_A]_{\mathcal{V}}$ is a diagonal matrix. Apply Theorem 8 with $\Phi \;=\; \Phi_A$, $\mathcal{A} \;=\; \mathcal{U}$, and $\mathcal{B} \;=\; \mathcal{V}$.$\blacksquare$

Proof of Theorem 11. If $\dim V = 1$, the result is trivial. Assume by induction that the result holds for vector spaces of dimension $n - 1$.

CASE 1. Suppose that $\Phi(\mathbf{x}, \mathbf{x}) \;=\; 0$ for all $\mathbf{x}$. Then $\Phi(\mathbf{x}, \mathbf{y}) \;=\; 0$ for all $\mathbf{x}$ and $\mathbf{y}$ because we have

$$\Phi(\mathbf{x}, \mathbf{y}) \;=\; \tfrac{1}{2}\Phi(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) \;-\; \Phi(\mathbf{x}, \mathbf{x}) \;-\; \Phi(\mathbf{y}, \mathbf{y})$$

and consequently $[\Phi]_{\mathcal{A}} \;=\; \mathbf{0}$ for *every* ordered basis $\mathcal{A}$.$\square$

CASE 2. Suppose that $\Phi(\mathbf{v}, \mathbf{v}) \;\neq\; 0$ for some $\mathbf{v}$. Let W be the set of all $\mathbf{x} \in V$ such that $\Phi(\mathbf{x}, \mathbf{v}) \;=\; 0$.[4] We claim that $W \;+\; \mathbb{F}\cdot\mathbf{v} \;=\; V$ and $W \cap \mathbb{F}\cdot\mathbf{v} \;=\; \{\mathbf{0}\}$. — The second assertion is trivial because $\Phi(\mathbf{v}, c\cdot\mathbf{v}) \;=\; 0$ implies that $c \cdot \Phi(\mathbf{v}, \mathbf{v}) \;=\; 0$. Since $\Phi(\mathbf{v}, \mathbf{v}) \;\neq\; 0$, this can only happen if $c \;=\; 0$, so that $c\cdot\mathbf{v} \;=\; \mathbf{0}$. To prove the first assertion, we must observe that for arbitrary $\mathbf{v} \in V$ the vector

$$\Pi(\mathbf{x}) \;=\; \mathbf{x} \;-\; \frac{\Phi(\mathbf{x}, \mathbf{v})}{\Phi(\mathbf{v}, \mathbf{v})}\,\mathbf{v}$$

[4] If Φ is the usual dot product on $\mathbb{R}^n$, then this is the hyperplane through $\mathbf{0}$ that is perpendicular to the line $\mathbf{0v}$.

lies in W (to verify this, compute $\Phi\big(\Pi(\mathbf{x}),\,\mathbf{v}\big)$ explicitly).[5] The conditions on W and $\mathbb{F}\cdot\mathbf{v}$ together with the dimension formulas imply that $\dim W \;=\; n-1$.

Consider the form Ψ obtained by restricting Φ to W; it follows immediately that Ψ is also symmetric. By the induction hypothesis there is a basis $\mathbf{w}_1,\ \cdots\ ,\mathbf{w}_{n-1}$ for W such that $\Phi(\mathbf{w}_i,\mathbf{w}_j) \;=\; 0$ if $i\neq j$. If we adjoin $\mathbf{v}$ to this set, then by the conditions on W and $\mathbb{F}\cdot\mathbf{v}$ we obtain a basis for V. Since $\Phi(\mathbf{v},\mathbf{w}_j)$ is zero for all j by the definition of W, it follows that the basis for V given by $\mathbf{v}$ together with $\mathbf{w}_1,\ \cdots\ ,\mathbf{w}_{n-1}$ will have the desired properties.■

The proof above actually gives and explicit method for finding a basis with the required properties: Specifically, start with a basis $\mathbf{v}_1,\ \cdots\ ,\mathbf{v}_n$ for V. If some $\mathbf{v}_i$ has the property $\Phi(\mathbf{v}_i\mathbf{v}_i)\neq 0$, rearrange the vectors so that the first one has this property. If $\Phi(\mathbf{v}_i,\mathbf{v}_i) \;=\; 0$ for all i, then either $\Phi \;=\; \mathbf{0}$ or else some value $\Phi(\mathbf{v}_i,\mathbf{v}_j)$ is nonzero (otherwise $\Phi=\mathbf{0}$ by Theorem 10). Rearrange the basis so that $\Phi(\mathbf{v}_1,\mathbf{v}_2)\neq 0$, and take a new basis $\{\ \mathbf{v}_i\ \}$ with $\mathbf{v}_1' \;=\; \mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{v}_i' \;=\; \mathbf{v}_i$ otherwise. Then $\Phi(\mathbf{v}_1',\mathbf{v}_1')\neq 0$, and thus in all cases we have modified the original basis to one having this property.

Now we modify $\mathbf{v}_i'$ such that $\mathbf{v}_1'' \;=\; \mathbf{v}_1'$ and $\Phi(\mathbf{v}_i'',\mathbf{v}_1'')=0$ if $i>1$. Specifically, if $i\geq 2$ let

$$\mathbf{v}_i'' \;=\; \mathbf{v}_i' \;-\; \frac{\Phi(\mathbf{v}_i',\mathbf{v}_1')}{\Phi(\mathbf{v}_1',\mathbf{v}_1')}\,\mathbf{v}_1'\ .$$

Having done this, we repeat the construction for $\mathbf{w}_1,\ \cdots\ ,\mathbf{w}_{n-1}$ for W with $\mathbf{w}_i=\mathbf{v}_{i+1}''$. When computing explicit numerical examples, it is often convenient to "clear the denominator of fractions" and multiply $\mathbf{v}_i''$ by $\Phi(\mathbf{v}_1',\mathbf{v}_1')$. This is particularly true when the matrix $\Phi(\mathbf{v}_i,\mathbf{v}_j)$ are integers (as in Exercise 2 below).

EXERCISES

1. Prove that the map sending bilinear forms to matrices in Theorem 7 is surjective.

2. Find an invertible matrix P such that ${}^{\mathbf{T}}PAP$ is diagonal, where A is the each of the following matrices with real entries:

$$\begin{pmatrix}1&0&1\\0&0&1\\1&1&1\end{pmatrix}\qquad\begin{pmatrix}1&0&1\\0&1&1\\1&1&2\end{pmatrix}\qquad\begin{pmatrix}2&1&3\\1&0&1\\3&1&1\end{pmatrix}\qquad\begin{pmatrix}0&1&0\\0&1&1\\0&1&1\end{pmatrix}\ .$$

3. A symmetric bilinear form Φ on an n-dimensional vector space V over a field $\mathbb{F}$ is said to be *nondegenerate* if for each nonzero $\mathbf{x}\in V$ there is some $\mathbf{y}\in V$ such that $\Phi(\mathbf{x},\mathbf{y})\neq 0$. Given an ordered basis $\mathcal{A}$ for V, show that Φ is nondegenerate if and only if the matrix $[\Phi]_{\mathcal{A}}$ is invertible. [*Hint:* Suppose that $\mathbf{x}$ satisfies $B\mathbf{x} \;=\; \mathbf{0}$, where B is the matrix in the previous sentence, and let $\mathbf{v} \;=\; \sum_i\ x_i\mathbf{a}_i$. If $\mathbf{w} \;=\; \sum_j\ y_j\mathbf{z}_j$, explain why ${}^{\mathbf{T}}\mathbf{y}B\mathbf{x} \;=\; \Phi(\mathbf{x},\mathbf{y})$ and how this is relevant.]

[5] If Φ is the ordinary dot product, then $\Pi(\mathbf{x})$ is the foot of the perpendicular dropped from $\mathbf{x}$ to the plane determined by W, and hence $\mathbf{0}\Pi(\mathbf{x})$ is perpendicular to W.

4. Projective classification of hyperquadrics

A standard exercise in plane and solid analytic geometry is the classification of conics and quadrics up to changes of coordinates given by rotations, reflections and translations. Stated differently, the preceding is the classification up to finding a rigid motion sending on to the other. An account of the classification for arbitrary dimensions appears on pages 257–262 of Birkhoff and MacLane, *Survey of Modern Algebra* (3rd Ed.). A related classification (up to finding an affine transformation instead of merely a rigid motion) is discussed in Exercise 4 below. In this section we are interested in the corresponding projective problem involving projective hyperquadrics and (projective) collineations.

Throughout this section we assume that $\mathbb{F}$ is a field in which $1+1 \neq 0$. Furthermore, if $\Sigma \subset \mathbb{FP}^n$ is a hyperquadric, then we shall use $\mathsf{SingSet}(\Sigma)$ to denote its subset of singular points.

We shall begin with an important observation.

THEOREM VII.12. *Let g be a projective collineation of $\mathbb{FP}^n$. Then a subset $\Sigma \subset \mathbb{FP}^n$ is a hyperquadric if and only if $g[\Sigma]$ is. Furthermore, the singular sets of these hyperquadrics satisfy*

$$g\,[\,\mathsf{SingSet}(\Sigma)\,] \;=\; \mathsf{SingSet}\,(\,T[\Sigma]\,)$$

and if $\mathrm{Tang}_X(\Sigma)$ *denotes the tangent hyperplane to Σ at a nonsingular point X, then*

$$g\,[\,\mathrm{Tang}_X(\Sigma)\,] \;=\; \mathrm{Tang}_{g(X)}\,(\,T[\Sigma]\,)\ .$$

Proof. Let A be a symmetric $(n+1)\times(n+1)$ matrix which defines the hyperquadric Σ.

According to Theorem VI.14, there is an invertible linear transformation C of $\mathbb{F}^{n+1,1}$ such that $T\,(\mathbb{F}\cdot\xi) \;=\; \mathbb{F}\cdot C(\xi)$ for all nonzero vectors $\xi \in \mathbb{F}^{n+1,1}$. Let B be the matrix of C in the standard basis. Then X lies in $T[\Sigma]$ if and only if $T^{-1}(X)$ lies in Σ. If ξ is a set of homogeneous coordinates for X, then the conditions in the preceding sentence are equivalent to

$${}^{\mathbf{T}}\xi\,{}^{\mathbf{T}}B^{-1}AB^{-1}\xi \;-\; 0$$

and the displayed equation is equivalent to saying that X lies on the hyperquadric associated to the (symmetric) matrix ${}^{\mathbf{T}}B^{-1}AB^{-1}$.

To check the statement about singular points, note that a point X lies on $\mathsf{SingSet}(\Sigma)$ if and only if X has homogeneous coordinates ξ such that ${}^{\mathbf{T}}\xi A \;=\; \mathbf{0}$, and the latter is equivalent to

$${}^{\mathbf{T}}\xi\,{}^{\mathbf{T}}B\,{}^{\mathbf{T}}B^{-1}AB^{-1} \;=\; \mathbf{0}$$

which in turn is equivalent to

$${}^{\mathbf{T}}(B\xi)\cdot\left({}^{\mathbf{T}}B^{-1}AB^{-1}\right) \;=\; \mathbf{0}\ .$$

To check the statement on tangent hyperplanes, note that Y lies on the tangent hyperplane to Σ at X if and only if there are homogeneous coordinates ξ for X and η for Y such that ${}^{\mathbf{T}}\xi A\eta \;=\; \mathbf{0}$, and the latter is equivalent to

$${}^{\mathbf{T}}\xi\,{}^{\mathbf{T}}B\,{}^{\mathbf{T}}B^{-1}AB^{-1}B\eta \;=\; \mathbf{0}$$

which in turn is equivalent to

$${}^{\mathbf{T}}(B\xi)\cdot\left(tp\,B^{-1}AB^{-1}\right)\eta) \;=\; \mathbf{0}\ .$$

The latter is equivalent to saying that $T(Y)$ is in the tangent hyperplane to $T[\Sigma]$ at $T(X)$.■

Definition. Two hypequadrics Σ andΣ' are *projectively equivalent* if there is a projective collineation T such that $T[\Sigma] = \Sigma'$. We sometimes write this relation as $\Sigma \sim \Sigma'$. It is clearly an equivalence relation, and the main goal of this section is to understand this relation when $\mathbb{F}$ is the real or complex numbers.

We shall first describe some necessary and sufficient conditions for the projective equivalence of hyperquadrics.

THEOREM VII.13. *Let Σ be a hyperquadric in $\mathbb{FP}^n$, and let T be a projective collineation of $\mathbb{FP}^n$. Then the following hold:*

(i) The dimensions of the geometrical subspaces of singular points of Σ and $T[\Sigma]$ must be equal.

(ii) If Σ contains no geometrical subspace of dimension r, then neither does $T[\Sigma]$.

Proof. (i) By definition, $\mathsf{SingSet}(\Sigma)$ is the set of all X whose homogeneous coordinates ξ satisfy ${}^{\mathbf{T}}\xi A = \mathbf{0}$, and hence $\mathsf{SingSet}(\Sigma)$ is a geometrical subspace. Now Theorem 12 implies that $T[\mathsf{SingSet}(\Sigma)] = \mathsf{SingSet}T[\Sigma]$, and hence

$$\dim \mathsf{SingSet}(\Sigma) \;=\; \dim T[\mathsf{SingSet}(\Sigma)] \;=\; \dim \mathsf{SingSet}T[\Sigma] \;.\square$$

(ii) Suppose $Q \subset T[\Sigma]$ is an r-dimensional geometrical subspace. Since T^{-1} is also a projective collineation, the set

$$T^{-1}[Q] \;\subset\; T^{-1}\left[\,T[\Sigma]\,\right] \;=\; \Sigma$$

is also an r-plane.■

THEOREM VII.14. *Suppose that Σ and Σ' are hyperquadrics which are defined by the symmetric matrices A and B respectively. Assume that there is an invertible matrix C and a nonzero constant k such that $B = {}^{\mathbf{T}}CAC$. Then Σ and Σ' are projectively equivalent.*

Proof. Let T be the projective collineation defined by C^{-1}, and if $X \in \mathbb{FP}^n$ let ξ be a set of homogeneous coordinates for X. Then by Theorem 12 we have

$$T[\Sigma] \;=\; \{\,\mathbf{x} \mid {}^{\mathbf{T}}\xi{}^{\mathbf{T}}CAC\xi \;=\; 0\,\} \;=\; \{\,\mathbf{x} \mid {}^{\mathbf{T}}\xi(k^{-1}B)\xi \;=\; 0\,\} \;=\; \{\,\mathbf{x} \mid k^{-1}\left({}^{\mathbf{T}}\xi B\xi\right) \;=\; 0\,\} \;=\; \Sigma' \;.■$$

NOTATION. Let D_r be the $n \times n$ diagonal matrix $(n \geq r)$ with ones in the first r entries and zeros elsewhere, and let $D_{p,q}$ denote the $n \times n$ diagonal matrix $(n \geq p+q)$ with ones in the first entries, (-1)'s in the next q entries, and zeros elsewhere.

REMARKS. 1. If A is a symmetric matrix over the complex numbers, then for some invertible matrix P the product ${}^{\mathbf{T}}PAP$ is D_r for some r. For Theorem 10 guarantees the existence of an invertible matrix P_0 such that $A_1 = {}^{\mathbf{T}}P_0AP_0$ is diagonal. Let P_1 be the diagonal matrix whose entries are square roots of the corresponding nonzero diagonal entries of A_1, and ones in the places where A_1 has zero diagonal entries. Then the product $P = P_0P_1^{-1}$ has the desired properties. This uses the fact that every element of the complex numbers $\mathbb{C}$ has a square root in $\mathbb{C}$, and in fact the same argument works in every field $\mathbb{F}$ which is closed under taking square roots.

2. If A is a symmetric matrix over the complex numbers, then for some invertible matrix P the product ${}^{\mathbf{T}}PAP$ is $D_{p,q}$ for some p and q. As in the preceding example, choose an invertible matrix P_0 such that $A_1 = {}^{\mathbf{T}}P_0AP_0$ is diagonal. Let P_1 be the diagonal matrix whose entries are square roots of the **absolute values** of the corresponding nonzero diagonal entries of A_1, and ones in the places where A_1 has zero diagonal entries. Then the product $P = P_0P_1^{-1}$ has the desired properties. The need for more complicated matrices arises because over $\mathbb{R}$ one only has square roots of nonnegative numbers, and if $x \in \mathbb{R}$ then either x or $-x$ is nonnegative.

The preceding remarks and Theorems 12–14 yield a complete classification of hyperquadrics in $\mathbb{FP}^n$ up to projective equivalence if $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. We shall start with the complex case, which is easier.

THEOREM VII.15. *Let $\Gamma_r \subset \mathbb{CP}^n$ defined by the matrix D_r described above. Then every nonempty hyperquadric in $\mathbb{CP}^n$ is projectively equivalent to Γ_r for some uniquely determined value of r.*

Proof. By Remark 1 above and Theorem 14, we know that Σ is projectively equivalent to Γ_r for some r. It suffices to show that if Γ_r and Γ_s are projectively equivalent then $r = s$.

By the preceding results we know that $\dim \mathsf{SingSet}(\Gamma_r)$ is the dimension of the subspace of all X whose homogeneous coordinates ξ satisfy ${}^{\mathbf{T}}\xi D_r = \mathbf{0}$, and the dimension of that subspace is equal to $n - r + 1$. Therefore, if Γ_r and Γ_s are projectively equivalent then we must have $n - r + 1 = n - s + 1$, which implies that $r = s$, so there is only one such hyperquadric that can be equivalent to Σ and thus uniqueness follows.■

The preceding argument goes through if $\mathbb{C}$ is replaced by an arbitrary field $\mathbb{F}$ which is closed under taking square roots.

Over the real numbers, the classification is somewhat more complicated but still relatively simple.

THEOREM VII.16. *Let $\Gamma_{p,q} \subset \mathbb{RP}^n$ defined by the diagonal matrix $D_{p,q}$ described above. Then every nonempty hyperquadric in $\mathbb{RP}^n$ is projectively equivalent to $\Gamma_{p,q}$ for some uniquely determined values of p and q such that $p \geq q$.*

Proof. As in the proof of the preceding theorem, by Theorem 14 and Remark 1 we know that an arbitrary projective hyperquadric is projectively equivalent to $\Gamma_{p,q}$ for some p and q. This hyperquadric is represented by $D_{p,q}$; if we permute the homogeneous coordinates, we see that $\Gamma_{p,q}$ is projectively equivalent to the hyperquadric defined by the matrix $-D_{q,p}$, and since the negative of this matrix defines the same hyperquadric it follows that $\Gamma_{p,q}$ is projectively equivalent to $\Gamma_{q,p}$. Since either $p \geq q$ or $q \geq p$, it follows that every hyperquadric is projectively equivalent to $\Gamma_{u,v}$ for some $u \geq v$.

To complete the proof, it will suffice to show that if $\Gamma_{p,q}$ is projectively equivalent to $\Gamma_{u,v}$ where $p \geq q$ and $u \geq v$, then $p+q = u+v$ and $p = u$. To see the first equality, note that the dimension of $\mathsf{SingSet}(\Gamma_{a,b})$ is equal to $n - (a + b) + 1$ by the argument in the preceding theorem, and as in that proof we conclude that $p + q = u + v$.

To see the second equality, we shall characterize the integer p in $\Gamma_{p,q}$ as follows.

(‡) *The hyperquadric $\Gamma_{p,q}$ contains a geometric subspace of dimension $n-p$ but no such subspace of higher dimension.*

This and the second part of Theorem 13 will combine to prove that if $\Gamma_{p,q}$ is projectively equivalent to $\Gamma_{u,v}$ where $p \geq q$ and $u \geq v$, then we also have $p = u$.

An explicit geometrical subspace S of dimension $N-p$ is given by the equations

$$\begin{array}{ll} x_i - x_{p+i} = 0 & 1 \leq i \leq q, \\ x_i = 0 & q \leq i \leq p \ . \end{array}$$

Consider the geometrical subspace T defined by

$$x_{p+1} = x_{p+2} = \cdots = x_{n+1} = 0 \ .$$

This geometrical subspace is $(p-1)$-dimensional. Furthermore, if $X \in T \cap \Sigma$ has homogeneous coordinates $(x_1, \cdots, x_{n+1})$ we have $x_i = 0$ for $i > p$, so that

$$\sum_{i \leq p} x_i^2 = 0 \ .$$

The latter implies that $x_i = 0$ for $i \leq p$, and hence it follows that $x_i = 0$ for **all** i; this means that *the intersection $T \cap \Sigma$ is the empty set.*

Suppose now that $S' \subset \Sigma$ is a geometrical subspace of dimension $\geq n-p+1$. Then the addition law for dimensions combined with $\dim(S' \star T) \leq n$ shows that $S' \cap T \neq \varnothing$, and since $S' \subset \Sigma$ we would also have $\Sigma \cap T \neq \varnothing$. But we have shown that the latter intersection is empty, and hence it follows that Σ cannot contain a geometrical subspace of dimension greater than $(n-p)$, which is what we needed to show in order to complete the proof.■

COMPUTATIONAL TECHNIQUES. Over the real numbers, there is another standard method for finding an equivalent hyperquadric defined by a diagonal matrix. Specifically, one can use the following diagonalization theorem for symmetric matrices to help find a projective collineation which takes a given hyperquadric to one of the given type:

Let A be a symmetric matrix over the real numbers. Then there is an orthogonal matrix P (one for which ${}^{\mathbf{T}}P = P^{-1}$) such that ${}^{\mathbf{T}}PAP$ is a diagonal matrix. Furthermore, if λ_i is the i^{th} entry of the diagonal matrix, then the i^{th} column of P is an eigenvector of A whose associated eigenvalue is equal to λ_i.■

This statement is often called the **Fundamental Theorem on Real Symmetric Matrices**, and further discussion appears on pages 51–52 of the following online notes:

`http://math.ucr.edu/~res/math132/linalgnotes.pdf`

If we combine the Fundamental Theorem on Real Symmetric Matrices with other material from this section, we see that the construction of a projective collineation taking the hyperquadric Σ_A defined by A to a hyperquadric defined by an equation of the form

$$\Sigma_i \, d_i x_i^2 = 0$$

reduces to finding the eigenvalues and eigenvectors of A. This approach is probably the most effective general method for solving problems like those in Exercise 3 below.

SPECIALIZATION TO THE REAL PROJECTIVE PLANE. We shall conclude this section by restating a special case of Theorem 16 that plays a crucial role in Section 6.

THEOREM VII.17. *All nonempty nonsingular conics in $\mathbb{RP}^2$ are projectively equivalent. In fact, they are equivalent to the affine unit circle which is defined by the homogeneous coordinate equation* $x_1^2 + x_2^2 - x_3^2 = 0$.

Proof. We must consider all $\Gamma_{p,q}$ with $p \geq q$ and $p + q = 3$ (this is the condition for the singular set to be empty). The only possibilities for (p, q) are $(2, 1)$ and $(3, 0)$. However, $\Gamma_{3,0}$ — the set of points whose homogeneous coordinates satisfy $x_1^2 + x_2^2 + x_3^2 = 0$ — is empty, so there is a unique possibility and it is given by $\Gamma_{2,1}$, which is the affine unit circle.■

EXERCISES

1. For each projective quadric in Exercise VII.1.3, determine the quadric in $\mathbb{RP}^3$ to which it is projectively equivalent.

2. Show that the number of projective equivalence classes of hypequadrics in $\mathbb{RP}^n$ is equal to $\frac{1}{4}(n+2)(n+4)$ if n is even and $\frac{1}{4}(n+3)^2$ if n is odd.

3. For each of the examples below, find a projective collineation of $\mathbb{RP}^2$ that takes the projectivizations of the following affine conics into the unit circle (with affine equation $x^2 + y^2 = 1$).

(i) The hyperbola $xy = 4$.

(ii) The parabola $y = x^2$.

(iii) The ellipse $4x^2 + 9y^2 = 36$.

(iv) The hyperbola $4x^2 - 9y^2 = 36$.

4. (a) What should it mean for two **affine** hyperquadrics in $\mathbb{R}^n$ to be affinely equivalent?

(b) Prove that every affine hyperquadric in $\mathbb{R}^n$ is equivalent to one defined by an equation from the following list:

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 = 0 \qquad (r \leq n)$$

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 + 1 = 0 \qquad (r \leq n)$$

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 + x_{r+1} = 0 \qquad (r < n)$$

See Birkhoff and MacLane, pp. 261–264, or Section V.2 of the online notes

`http://math.ucr.edu/~res/math132/linalgnotes.pdf`

for further information on this topic.

5. Duality and projective hyperquadrics

In this section we shall show that the duality properties for geometrical subspaces of coordinate projective spaces are part of a larger pattern of duality which includes hyperquadrics. As in most other sections of these notes, $\mathbb{F}$ will denote a (commutative) field in which $1 + 1 \neq 0$.

Definition. A *hypersurface of the second class* in $(\mathbb{FP}^n)^*$ is the set $\mathfrak{F}$ of all hyperplanes H whose homogeneous coordinates θ satisfy a homogeneous quadratic equation

$$\theta B^{\mathbf{T}}\theta \;=\; 0.$$

If we write $\theta \;=\; (u_1, \;\cdots\;, u_{n+1})$ and B has entries $b_{i,j}$, this is equivalent to the scalar equation $\sum_{i,j} \; b_{i,j} u_i u_j \;=\; 0$.

The dual of a tangent line is a *cotangent* $(n-2)$*-subspace* V contained in a hyperplane H which belongs to the hypersurface $\mathfrak{F}$. The cotangent condition becomes an assertion that *exactly one of the following two statements is valid*:

(i) H is the only hyperplane in $\mathfrak{F}$ containing V.

(ii) Every hyperplane in $\mathfrak{F}$ contains V.

In the first case we say that H is a *nonsingular* hyperplane in $\mathfrak{F}$, and in the second we say that H is a *singular* hyperplane in $\mathfrak{F}$.

By duality the set of all cotangent $(n-2)$-spaces at a nonsingular hyperplane is the set of all $(n-2)$-spaces through a point called the *point of contact* of $\mathfrak{F}$ at H. This point has homogeneous coordinates $B^{\mathbf{T}}\theta$.

Nonsingular hyperquadrics and nonsingular hypersurfaces of the second class satisfy the following useful interrelationship:

THEOREM VII.18. (i) *The set of all hyperplanes* $\mathbf{T}\Sigma$ *tangent to a nonsingular hyperquadric* Σ *is a hypersurface of the second class.*

(ii) *The set of all points of contact* $\mathbf{K}\mathfrak{F}$ *to a nonsingular hypersurface* $\mathfrak{F}$ *of the second class is a nonsingular hyperquadric.*

(iii) *In the setting of the preceding two statements, we have* $\mathbf{KT}\Sigma \;=\; \Sigma$ *and* $\mathbf{TK}\mathfrak{F} \;=\; \mathfrak{F}$.

Proof. (i) Suppose Σ is defined as the set of all X whose homogeneous coordinates satisfy ${}^{\mathbf{T}}\xi A\xi \;=\; 0$. We claim that H lies in $\mathbf{T}\Sigma$ if and only if its homogeneous coordinates θ satisfy $\theta A^{-1}\,{}^{\mathbf{T}}\theta \;=\; 0$.

Suppose that $H \in \mathbf{T}\Sigma$. Let X be a point such that H is the tangent hyperplane to X, and let θ and ξ be homogeneous coordinates for H and X respectively. Then we have $\theta \;=\; {}^{\mathbf{T}}\xi A$, and hence

$$\theta A^{-1\,\mathbf{T}}\theta \;=\; \left({}^{\mathbf{T}}\xi A\right)\theta A^{-1}(A\xi) \;=\; {}^{\mathbf{T}}\xi A\xi \;=\; 0$$

which is what we wanted to prove.

Conversely, suppose that homogeneous coordinates θ for H satisfy the equation

$$\theta A^{-1}\,{}^{\mathbf{T}}\theta \;=\; 0\;.$$

Let $\xi = A^{-1\,\mathbf{T}}\theta$, and let X have homogeneous coordinates ξ. Then $\theta = {}^{\mathbf{T}}\xi A$ and

$$ {}^{\mathbf{T}}\xi A\xi \;=\; {}^{\mathbf{T}}\xi A\, A^{-1}\, A\xi \;=\; \theta A^{-1}\, {}^{\mathbf{T}}\theta \;=\; 0 $$

so that H lies in $\mathbf{T}\Sigma$.

Finally, note that $\mathbf{T}\Sigma$ is nonsingular because it is defined by the invertible matrix A^{-1}.$\Box$

(ii) The statement about $\mathbf{K}\mathfrak{F}$ follows by duality. It remains to show that $\mathbf{KT}\Sigma = \Sigma$. Howver, if Σ is defined by the invertible matrix A, then $\mathbf{T}\Sigma$ is defined by the inverse matrix A^{-1}, and therefore by duality it follows that $\mathbf{KT}\Sigma$ is defined by the matrix

$$ \left(A^{-1}\right)^{-1} \;=\; A $$

and hence it must be equal to Σ.

Finally, the assertion $\mathbf{TK}\mathfrak{F} = \mathfrak{F}$ follows by duality.$\blacksquare$

Extending duality to nonsingular hyperquadrics. The preceding theorem implies the following general principle:

AUGMENTED PRINCIPLE OF DUALITY. *A statement about coordinate projective n-spaces over fields remains true if — in addition to the previously specified interchanges involving geometrical subspaces — one interchanges the phrases* point on a nonsingular hyperquadric *and* tangent hyperplane to a nonsingular hyperquadric.$\blacksquare$

Important examples of this extended dualization will be given in the next (and final) section of these notes.

EXERCISES

1. Find the equations defining the tangent lines to the projectivizations of the following affine conics:

(i) The parabola $y^2 = 4ax$.

(ii) The ellipse $a^2x^2 + b^2y^2 = a^2b^2$.

(iii) The hyperbola $a^2x^2 - b^2y^2 = a^2b^2$.

(iv) The hyperbola $xy = a$.

2. Find the equation defining the conic in $\mathbb{RP}^2$ whose tangent lines satisfy the equation

$$ u_1^2 \;-\; 2u_1u_2 \;+\; u_2^2\, 2u_2u_3 \;+\; 2u_1u_3 \;+\; u_3^2 \;=\; 0\;. $$

[*Hint:* Look at the proof of Theorem 18.]

3. Write out the plane dual to the following statements about conics in the projective plane $\mathbf{P}$:

(i) At the points X and Y on the nonsingular conic Γ, the respective tangent lines L and M meet at a point Z.

(ii) No three points of the nonsingular conic Γ are collinear.

(iii) There are two lines in the (projective) plane $\mathbf{P}$ that are tangent to both of the nonsingular conics Γ_1 and Γ_2.

6. Conics in the real projective plane

Projective conics have a great many interesting properties, the most famous of which is Pascal's Theorem (see Theorem 24 below). A thorough discussion of projective conics appears in Coolidge, *A History of the Conic Sections and Quadric Surfaces*. In this section we shall limit ourselves to proving a few of the more important and representative theorems in the subject.

Throughout this section we shall be considering coordinate projective planes over a fixed field $\mathbb{F}$ in which $1 + 1 \neq 0$. We shall also assume that $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_3$ after Theorem 22. Of course, this means that all the results in this section are valid in the real and complex projective planes.

THEOREM VII.19. *Given any five points in $\mathbb{FP}^2$, no three of which are collinear, there is a unique conic containing them. Furthermore, this conic is nonsingular.*

Proof. Let $A,\ B,\ C,\ D,\ E,\ V$ be five points, no three of which are collinear. We shall first prove the result in a special case and then prove that it holds more generally.

CASE 1. Suppose that homogeneous coordinates $\alpha,\ \beta,\ \gamma,\ \delta$ for $A,\ B,\ C,\ D$ are given by standard values:

$$\alpha = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \beta = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \gamma = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \delta = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

Since no three of the five points are collinear, homogeneous coordinates for V are given by $a_0\alpha+b_0\beta+c_0\gamma$, where none of $a_0,\ b_0,\ c_0$ is equal to zero. Dividing by a_0, we can find homogeneous coordinates ψ for V such that $\psi = \alpha + b\beta + c\gamma$. Since $V \notin AD$, the scalars b and c must be distinct.

Suppose there is a conic Γ containing $A,\ B,\ C,\ D,\ E,\ V$ and that it is defined by the symmetric 3×3 matrix P. We need to determine the entries $p_{i,j}$ from the equations ${}^{\mathbf{T}}\xi P \xi = 0$, which should hold for $\xi - \alpha,\ \beta,\ \gamma,\ \delta,\ \psi$. If $\xi = \alpha$, then direct substitutions implies that $p_{1,1} = 0$. Likewise, if we make the substitutions $\xi = \beta$ and $\xi = \gamma$ we find that $p_{2,2} = p_{3,3} = 0$. Continuing in this manner, if we make the substitution $\xi = \delta$ and use the previously derived values for the diagonal entries, we find that

$$2 \cdot (p_{1,2} + p_{2,3} + p_{1,3}) = 0$$

and if we follow this with the substitution $\xi = \psi$ we also obtain the equation

$$2 \cdot (b\,p_{1,2} + bc\,p_{2,3} + c\,p_{1,3}) = 0 .$$

Thus the entries of the symmetric matrix P satisfy the following conditions:

$$p_{1,1} = p_{2,2} = p_{3,3} = 0$$

$$p_{1,2} = \frac{(1-b)c}{b-c}\,p_{2,3} \qquad\qquad p_{1,3} = \frac{(1-c)b}{c-b}\,p_{2,3}$$

Therefore the coefficients of P are uniquely determined up to a scalar multiple, and it follows that there is at most one conic containing the given five points. On the other hand, if we set

$$P = \begin{pmatrix} 0 & (1-b)c & b(c-1) \\ (1-b)c & 0 & b-c \\ b(c-1) & b-c & 0 \end{pmatrix}$$

then the preceding calculations show that the given five points lie in the conic defined by P.

To prove that the conic is nonsingular, it suffices to show that the determinant of the matrix P defined above is nonzero. First note that $V \notin CD$ and $V \notin BD$ imply $b \neq 1$ and $c \neq 1$ respectively. Thus the determinant of P, which is

$$2\,(1-b)c(b-c)b(b-1)$$

must be nonzero, proving that the conic is nonsingular.$\square$

Case 2. Suppose that $A,\ B,\ C,\ D,\ E,\ V$ are arbitrary. By the Fundamental Theorem of Projective Geometry, there is a projective collineation Φ such that

$$\Phi(A),\ \Phi(B),\ \Phi(C),\ \Phi(D),\ \Phi(E),\ \Phi(V)$$

satisfy the conditions of Case 1. Let Γ_0 be the unique nonsingular conic given by Case 1; then $\Gamma = \Phi^{-1}[\Gamma_0]$ is a nonsingular conic containing $A,\ B,\ C,\ D,\ E,\ V$ by Theorem 12, proving existence. To show uniqueness, suppose that Γ' is another conic containing the given five points; then $\Phi[\Gamma']$ is a conic containing $\Phi(A),\ \Phi(B),\ \Phi(C),\ \Phi(D),\ \Phi(E),\ \Phi(V)$ and therefore by Case 1 we have $\Phi[\Gamma'] = \Gamma_0$. Consequently, we have $\Gamma' = \Phi^{-1}\circ\Phi[\Gamma'] = \Phi^{-1}[\Gamma_0] = \Gamma$.$\blacksquare$

If $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_3$ then there is a converse to the preceding theorem; namely, every nonsingular conic contains at least five points (see Theorem 21). In fact, no three of these points can be collinear, for the noncollinearity of three arbitrary points on a nonsingular conic will be a consequence of the next result.

Theorem VII.20. *No three points on a nonsingular conic are collinear.*

Proof. Let $A,\ B,\ C$ be three collinear points. We claim that every conic containing all three of them is singular. By the Fundamental Theorem of Projective Geometry and Theorem 12, it suffices to consider the case in which homogeneous coordinates α and β for A and B are the first two unit vectors in $\mathbb{F}^{3,1}$.

By Theorem V.7, homogeneous coordinates γ for C may be chosen so that $\gamma = \alpha + c\gamma$, where $c \neq 0$. If the conic $\Gamma$is defined by the symmetric 3×3 matrix P, then computations like those of Theorem 19 imply that $p_{1,1} = p_{2,2} = 0$ and $2c\,p_{1,2} = 0$. Thus P has the following form:

$$P = \begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & p_{2,3} \\ p_{1,3} & p_{2,3} & p_{3,3} \end{pmatrix}$$

However, direct computation shows that such a matrix is not invertible, and therefore the conic Γ is singular by Theorem 5.$\blacksquare$

Here is the other result we need to establish a converse to Theorem 19:

Theorem VII.21. *Let Γ be a nonempty conic in $\mathbb{FP}^2$ containing at least one nonsingular point, and assume that the field $\mathbb{F}$ contains at least n distinct elements. Then Γ contains at least $(n+1)$ distinct points.*

In particular, if $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_3$, then Γ contains at least five distinct points (note that $\mathbb{F}$ cannot be isomorphic to $\mathbb{Z}_2$ because we are assuming that $1+1 \neq 0$ in $\mathbb{F}$).

Proof. Let $X \in \Gamma$ be a nonsingular point, and let L be the tangent line through X. Then there are at least n other lines through X, say $L_1, \cdots, L_n$. Since each L_i is not a tangent line and $X \in L_i \cap \Gamma$, there must be a second point $X_i \in L_i \cap \Gamma$.

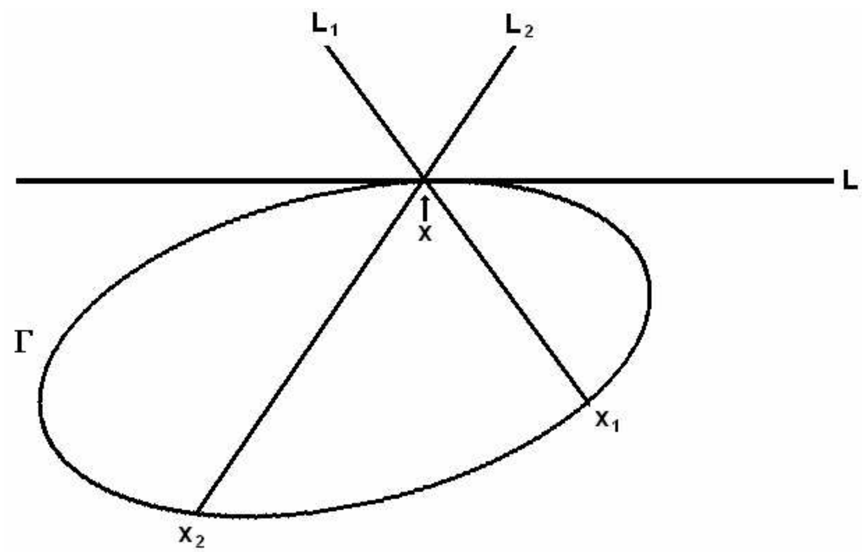

Figure VII.1

If $i \neq j$, then $X_i \neq X_j$ because otherwise L_i and L_j would have two points in common and we know these lines are distinct. Therefore the points $X, X_1, \cdots, X_n$ must be distinct points of Γ.■

A synthetic approach to conics

The theorem above give an incidence-theoretic characterization of nonsingular conics and suggest that synthetic methods might be useful in the study of conics. The next two theorems give a completely synthetic characterization of nonsingular conics due to J. Steiner.[6]

From this point on, unless stated otherwise, we shall assume that the field $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_3$.

Theorem VII.22. *Let A and B be distinct points in $\mathbb{FP}^2$, and let Φ be a projective collineation of $\mathbb{FP}^2$ sending A to B. Then*

$$\mathbf{K} = \{X \in \mathbb{FP}^2 \mid X = A \quad \text{or} \quad X \in \Phi(L) \cap L \text{ for some line } L \text{ through } A\}$$

is a conic. (Notice that $B \in \mathbf{K}$, for we may take $L = AB$ in the definition).

[6]Jakob Steiner (1796–1863) is known for his work on projective geometry from a strongly synthetic viewpoint and for results in other branches of geometry.

Proof. Let P be an invertible 3×3 such that if ξ is a set of homogeneous coordinates for X, then $\Phi(X) = \mathbb{F} \cdot P\xi$. Also, let α be a set of homogeneous coordinates for A, and let β be a set of homogeneous coordinates for B such that $\beta = P \cdot \alpha$.

We need to find a homogeneous quadratic equation which defines $\mathbf{K}$. By Exercise V.1.5, if L is a line in $\mathbb{FP}^2$ and has homogeneous coordinates λ, then the line $\Phi[L]$ has homogeneous coordinates λP^{-1}. Thus $X \in \mathbf{K}$ if and only if $X = A$ or its homogeneous coordinates ξ satisfy

$$\xi = {}^{\mathbf{T}}\big(\lambda P^{-1}\big) \times {}^{\mathbf{T}}\lambda$$

for some line L whose homogeneous coordinates λ satisfy $\lambda \cdot \alpha = 0$. Equivalently, we have $X \in \mathbf{K}$ if and only if $X = A$ or

$$\lambda P^{-1}\xi = \lambda \cdot \xi = \lambda \cdot \alpha = 0 .$$

It follows that $X \in \mathbf{K}$ if and only if α, ξ and $p^{-1}\xi$ are linearly dependent (the case $X \neq A$ is immediate from the preceding three equations, while the case $X = A$ is trivial). Since P defines an invertible linear transformation, the vectors α, ξ and $P^{-1}\xi$ are linearly independent if and only if $\beta = P \cdot \alpha$, $P \cdot \xi$ and $\xi = P\,P^{-1}\xi$ are linearly independent. The linear dependence of the latter is in turn equivalent to the vanishing of the determinant $[\xi,\ P\,\xi,\ \beta]$. But the latter expression is a homogeneous quadratic polynomial in the entries of ξ and hence it is the defining equation of a conic.■

Conversely, every nonsingular conic is defined by a projective collineation as in Theorem 22.

THEOREM VII.23. (Steiner) *Let Γ be a nonsingular conic in $\mathbb{FP}^2$ containing at least five distinct points, and let A and B be distinct points of Γ. Then there is a projective collineation Φ of $\mathbb{FP}^2$ sending A to B such that*

$$\Gamma = \{X \in \mathbb{FP}^2 \mid X = A \quad \text{or} \quad X \in \Phi(L) \cap L \text{ for some line } L \text{ through } A\} .$$

Proof. Let $X,\ Y,\ Z$ be three points of Γ which are distinct from A and B. By Theorem 20, no three of the points $A,\ B,\ X,\ Y,\ Z$ are collinear. Thus there is a unique projective collineation Φ sending A to B and $X,\ Y,\ Z$ to themselves. By Theorem 22, the points $A,\ B$ and the collineation Φ determine a conic Γ' defined by the formula above. By construction the three points $X,\ Y,\ Z$ lie on Γ', and therefore $\Gamma = \Gamma'$ by Theorem 12.■

NOTATION. If Γ is a conic and $A,\ B \in \Gamma$, then the collineation Φ of Theorem 23 is called a *Steiner collineation* associated to A, B and Γ. We note that this collineation is **not** unique, for different choices of the three points $X,\ Y,\ Z$ yield different collineations.

Conics and inscribed polygons

Definition. Let $P_1, \cdots, P_n$ be $n \geq 3$ points in $\mathbb{FP}^2$ such that no three are collinear. The **simple (projective) n-gon** $P_1 \cdots P_n$ is defined to be

$$P_1P_2 \cup \cdots \cup P_{n-1}P_n \cup P_nP_1 .$$

Dually, if $L_1, \cdots, L_n$ is a set of $n \geq 3$ lines such that no three are concurrent, the dual of a simple n-gon is the finite set of points determined by the intersections $L_i \cap L_{i+1}$ and $L_n \cap L_1$ (*i.e.*, a set of n points such that no three are collinear), and the union of the lines is the simple n-gon determined by these n points.

The following result due to B. Pascal[7] is one of the most celebrated theorems in projective geometry:

THEOREM VII.24. (Pascal's Theorem) *Suppose that Γ is a nonsingular conic in $\mathbb{FP}^2$ and let the simple hexagon $A_1 \cdots A_6$ be* inscribed *in Γ (in other words, $A_i \in \Gamma$ for all i). Let*

$$X = A_1A_2 \cap A_4A_5 , \qquad Y = A_2A_3 \cap A_5A_6 , \qquad Z = A_3A_4 \cap A_6A_1 .$$

Then X, Y and Z are collinear.

The line containing these three points is called the *Pascal line* of the hexagon.

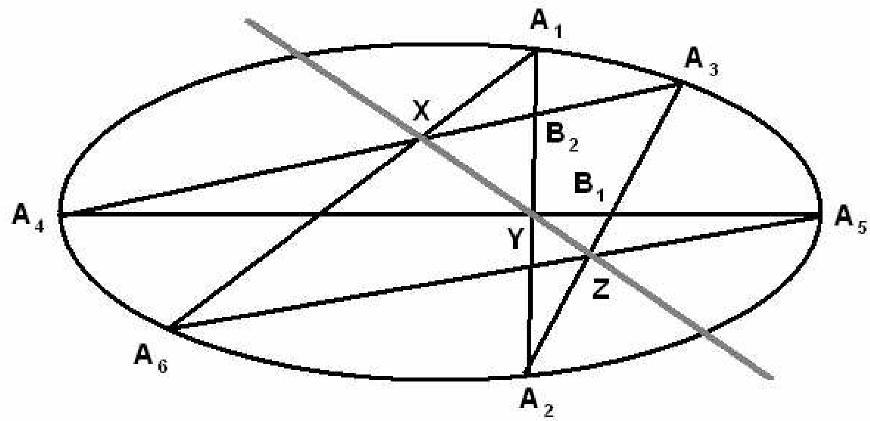

Figure VII.2

We have stated Pascal's Theorem for nonsingular conics, but a version of the result is also true for singular conics given by the union of two lines, provided the hexagon is degenerate in the sense that $\{A_1, A_3, A_5\}$ lie on one line and $\{A_2, A_4, A_6\}$ lie on the other. In such a situation, *the conclusion of Pascal's Theorem reduces to the conclusion of Pappus' Theorem*, and hence one can view Pappus' Theorem as a special case of Pascal's Theorem.[8]

SPECIAL CASE. Suppose that Γ in $\mathbb{RP}^2$ is given by the ordinary unit circle and $A_1 \cdots A_6$ is a regular hexagon which is inscribed in Γ. Then it is clear that $A_1A_2||A_41A_5$, $A_2A_3||A_51A_6$ and $A_3A_4||A_61A_1$ (see the illustration below — note that A_1, A_2, A_3, A_4, A_5, A_6 correspond to A, B, C, D, E, F in the drawing), so that X, Y and Z are all ideal points and the Pascal line is equal to the line at infinity.

[7]BLAISE PASCAL (1623–1662) is known for contributions to a wide range of areas in the mathematical and physical sciences as well as philosophy. Aside from the theorem appearing here, he is particularly recognized for scientific work on fluid mechanics, probability theory, a counting machine which was the prototype for devices like mechanical odometers, as well as the philosophy of science. Most of his philosophical writings were highly religious in nature.

[8]And this is why French and German writers often use phrases translating to "Pascal's Theorem" when referring to the result known as Pappus' (Hexagon) Theorem in the English language.

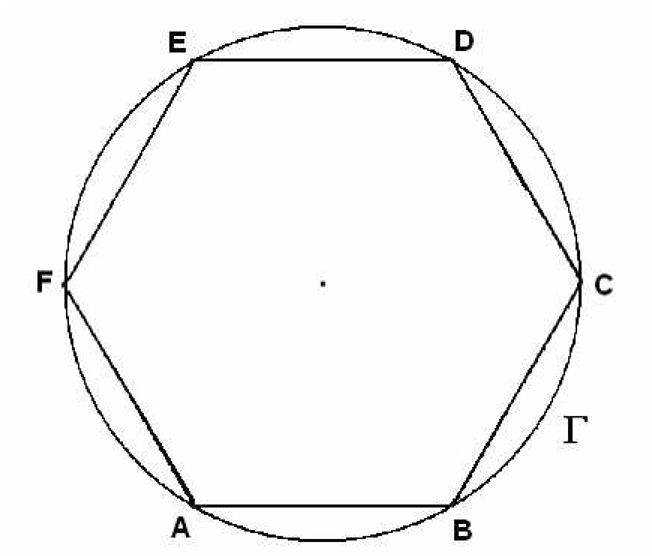

Figure VII.3

Proof of Pascal's Theorem (= Theorem 24). By Steiner's Theorem there is a projective collineation Φ such that $\Phi(A_1) = A_5$ and Φ also has the following properties:

$$\Phi[A_1A_4] = A_5A_4 \qquad \Phi[A_1A_2] = A_5A_2 \qquad \Phi[A_1A_3] = A_5A_3$$

$$\Phi[A_1Z = A_1A_6] = A_5A_6 = A_5Y$$

As suggested by Figure VII.2, we define B_1 to be the point at which A_2A_3 meets A_4A_5, and we define B_2 to be the point at which A_3A_4 meets A_1A_2. Since Φ is a projective collineation, by Exercise VI.3.3 we have the following cross ratio equations:

$$\mathsf{XR}\,(A_1A_4,\ A_1A_3,\ A_1B_2,\ A_1Z) = \mathsf{XR}\,(\ \Phi[A_1A_4],\ \Phi[A_1A_3],\ \Phi[A_1B_2],\ \Phi[A_1Z]\ ) =$$
$$\mathsf{XR}\,(A_5B_1,\ A_5A_3,\ A_5A_4,\ A_5Y)$$

By construction, the points Z and B_2 are on A_3A_4, and the points Y and B_1 are on $A-2A_3$. Therefore Theorem V.14 implies that the first cross ratio in the displayed equation is equal to $\mathsf{XR}(A_4, A_3, B_2, Z)$ and the second is equal to $\mathsf{XR}(B_1, A_3, A_2, Y)$, so that

$$\mathsf{XR}(A_4, A_3, B_2, Z) = \mathsf{XR}(B_1, A_3, A_2, Y)\ .$$

Since $A_4B_1 = A_4A_5$ and $B_2A_2 = A_1A_2$ it follows that $X \in A_4B_1 \cap A_3X \cap B_2A_2$. Thus we also have

$$\mathsf{XR}(B_1, A_3, A_2, Y) = \mathsf{XR}(A_4, A_3, B_2, W)$$

where $W \in A_3A_4 \cap XY$. But the right hand side of the equation is also equal to the cross ratio $\mathsf{XR}(A_4, A_3, B_2, W)$, and therefore $W = Z$ by Theorem V.10. In particular, this implies that

$$Z \in A_3A_4 \cap XY$$

so that X, Y and Z are collinear.■

If we now apply the Augmented Principle of Duality formulated in Section V, we immediately obtain the following result (Brianchon's Theorem),[9] which was originally established without using duality:

[9]Charles Julien Brianchon (1783–1864) worked in mathematics and chemistry; in mathematics he is known for rediscovering Pascal's Theorem and proving the result which bears his name.

THEOREM VII.25. (Brianchon's Theorem) *Suppose that* Γ *is a nonsingular conic in* $\mathbb{FP}^2$ *and let the simple hexagon* $A_1 \cdots A_6$ *be* circumscribed *about* Γ *(in other words, the lines* A_iA_{i+1} *are tangent to Gamma for all* i*, and likewise for* A_6A_1*). Then the lines* A_1A_4*,* A_2A_3 *and* A_2A_3 *are concurrent.*■

The point of concurrency is called the *Brianchon point.*
SPECIAL CASE. Suppose that Γ in $\mathbb{RP}^2$ is given by the ordinary unit circle and $A_1 \cdots A_6$ is a regular hexagon which is inscribed in Γ. Then the Brianchon point is the center of the circle.

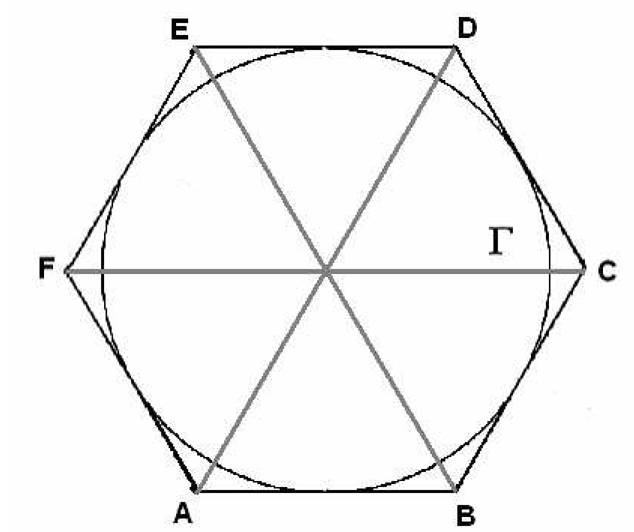

Figure VII.4

There is also a converse to Pascal's Theorem (and hence, by duality, there is also a converse to Brianchon's Theorem).

THEOREM VII.26. *Let* $A_1 \cdots A_6$ *be a simple hexagon, and let* X*,* Y*,* Z *be defined as in Pascal's Theorem. If these three points are collinear, then there is a conic* Γ *such that* $A_i \in \Gamma$ *for all* i*.*

By Theorem 19, if there is a conic containing the give six points, then it must be nonsingular.

Proof. Let Γ be the unique nonsingular conic containing the first five points $A_1, \cdots, A_5$ and let Φ be the Steiner collineation for Γ with $\Phi(A_1) = A_5$ and $\Phi[A_1A_j] = A_5A_j$ for $j = 2, 3, 4$. It will suffice to prove that $\Phi[A_1A_6] = A_5A_6$. Let B_1 and B_2 be defined as in the proof of Pascal's Theorem. Since $\{A_4, A_3, B_2, Z\}$ and $\{B_1, A_3, A_2, Y\}$ are quadruples of collinear points and

$$X \in A_4B_1 \cap A_3 \cap A_2B_2 \cap YZ$$

Theorem V.14 implies that

$$\mathsf{XR}(A_4, A_3, B_2, Z) = \mathsf{XR}(B_1, A_3, A_2, Y) .$$

This in turn implies the following equation:

$$\mathsf{XR}(A_1A_4, A_1A_3, A_1B_2, A_1Z) = \mathsf{XR}(A_5B_1, A_5A_3, A_5A_2, A_5Y)$$

Since $A_1B_2 = A_1A_2$, $A_1Z = A_1A_6$, $A_5B_1 = A_5A_4$ and $A_5Y = A_5A_6$, the equation above may be rewritten as follows:

$$\mathsf{XR}(A_1A_4, A_1A_3, A_1A_2, A_1A_6) = \mathsf{XR}(A_5A_4, A_5A_3, A_5A_2, A_5A_6)$$

On the other hand, since Φ is a projective collineation, the right hand side is equal to the following:

$$\mathsf{XR}\big(\,\Phi[A_1A_4],\,\Phi[A_1A_3],\,\Phi[A_1A_2],\,\Phi[A_1A_6]\,\big) \;=\; \mathsf{XR}(A_5A_4, A_5A_3, A_5A_2, \Phi[A_1A_6]\,)$$

Therefore it follows that $\Phi[A_1A_6] = A_5A_6$, which is what we needed to verify in order to complete the proof.■

The statement of the dual theorem to Theorem 26 is left to the reader (see the exercises).■

Degenerate cases of Pascal's Theorem

There are analogs of Pascal's Theorem for inscribed simple n-gons where $n = 3, 4, 5$ (and by duality there are similar analogs of Brianchon's Theorem). Roughly speaking, these are limiting cases in which two consecutive vertices merge into a single point and the line joining the two points converges to the tangent line at the common point. The proofs of these theorems require a simple observation about Steiner collineations.

THEOREM VII.27. *Let Γ be a nonsingular conic, let A and B be points of Γ, and let Φ be a Steiner collineation for Γ such that $\Phi(A) = B$. If $\mathbf{T}_A$ is the tangent line to Γ at A, then $\Phi[\mathbf{T}_A] = AB$; if $\mathbf{T}_B$ is the tangent line to Γ at B, then $\Phi[AB] = X\mathbf{T}_B$.*

Proof. Since $B \in \Phi[\mathbf{T}_A]$, we know that $\Phi[\mathbf{T}_A] = BC$ for some point C. If $D \in \Phi[\mathbf{T}_A] \cap BC$, then $D \in \Gamma$ by construction. But the only point in $\mathbf{T}_A \cap \Gamma$ is A itself, and therefore we must have $BC = BA$. Since Φ^{-1} is a Steiner collineation for Γ taking B to X, it follows that $\Phi^{-1}[\mathbf{T}_B] = AB$, which is equivalent to the desired equation $\Phi[AB] = X\mathbf{T}_B$.■

Here are the analogs of Pascal's Theorem for inscribed pentagons and quadrilaterals; note that there are two separate analogs for quadrilaterals.

THEOREM VII.28. *Suppose that Γ is a nonsingular conic in $\mathbb{FP}^2$ and let the simple pentagon $A_1 \cdots A_5$ be inscribed in Γ. Let*

$$X = A_1A_2 \cap A_4A_5\,, \qquad Y = A_2A_3 \cap A_5A_1\,, \qquad Z = A_3A_4 \cap \mathbf{T}_{A_1}\,.$$

Then X, Y and Z are collinear.

THEOREM VII.29. *Suppose that Γ is a nonsingular conic in $\mathbb{FP}^2$ and let the simple quadrilateral $A_1 \cdots A_4$ be inscribed in Γ. Let*

$$X = \mathbf{T}_{A_1} \cap A_2A_4\,, \qquad Y = A_1A_2 \cap A_3A_4\,, \qquad Z = \mathbf{T}_{A_2} \cap A_1A_4\,.$$

Then X, Y and Z are collinear.

THEOREM VII.30. *Suppose that Γ is a nonsingular conic in $\mathbb{FP}^2$ and let the simple quadrilateral $A_1 \cdots A_4$ be inscribed in Γ. Let*

$$D = A_1A_3 \cap A_2A_4\,, \qquad E = A_1A_4 \cap A_2A_3\,, \qquad F = \mathbf{T}_{A_1} \cap \mathbf{T}_{A_2}\,.$$

Then D, E and F are collinear.

The proofs of these theorems are easy variants of the proofs of Pascal's Theorem and are left to the reader as exercises.■

Similarly, formulations and proofs of the duals to all these results are left to the reader as exercises.■

The final degenerate case of Pascal's Theorem requires a special argument. As noted in Appendix A, the cross product of vectors in $\mathbb{F}^3$ satisfies the following condition known as the *Jacobi Identity* (see Theorem A.21):

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \;+\; \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) \;+\; \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) \;=\; \mathbf{0}\,.$$

THEOREM VII.31. *Let ABC be a (projective) triangle inscribed in the nonsingular conic Γ. Let*

$$D \;=\; \mathbf{T}_A \,\cap\, BC\;, \qquad E \;=\; \mathbf{T}_B \,\cap\, AC\;, \qquad F \;=\; \mathbf{T}_C \,\cap\, AB\;.$$

Then D, E and F are collinear.

Proof. By Theorems 5, 10 and 14, the nonsingular conic Γ is projectively equivalent to one defined by an equation of the form $ax^2+by^2+cz^2 \;=\; 0$ where none of the coefficients $a,\ b,\ c$ is equal to zero. Dividing these by a suitable constant, we may assume $c=-1$. Therefore it suffices to prove the result for conics defined by equations of the form $ax^2+by^2-z^2=0$.

Let ρ be the 1×3 matrix $(0\ 0\ -2)$. If $X\in\Gamma$ and has homogeneous coordinates

$$\xi \;=\; {}^{\mathbf{T}}(x_1 \quad x_2 \quad x_3)$$

then homogeneous coordinates for the tangent line $\mathbf{T}_X$ to Γ at X are given by $\xi^{\#} \;=\; {}^{\mathbf{T}}\xi \,+\, x_3\,\rho$.

Let α, β, γ denote homogeneous coordinates for $A,\ B,\ C$, and let $\alpha^{\#}$, $\beta^{\#}$, $\gamma^{\#}$ denote corresponding homogeneous coordinates for the tangent lines $\mathbf{T}_A$, $\mathbf{T}_B$ and $\mathbf{T}_C$. It will suffice to show that the vectors

$$\alpha^{\#} \;\times\; (\beta\times\gamma)\;, \qquad \beta^{\#} \;\times\; (\gamma\times\alpha)\;, \qquad \gamma^{\#} \;\times\; (\alpha\times\beta)$$

are linearly dependent. However, their sum is equal to

$$[\alpha \;\times\; (\beta\times\gamma) \;+\; \beta \;\times\; (\gamma\times\alpha) \;+\; \gamma \;\times\; (\alpha\times\beta)] \;+$$
$${}^{\mathbf{T}}\rho \;\times\; \big(a_3\,\beta\times\gamma \;+\; b_3\,\gamma\times\alpha \;+\; c_3\,\alpha\times\beta\big)$$

and we claim that this sum vanishes. The term in square brackets vanishes by the Jacobi Identity; to analyze the remaining term(s), we may use the "back–cab formula"

$${}^{\mathbf{T}}\rho \;\times\; (\eta\times\zeta) \;=\; (\rho\cdot\zeta)\eta \;-\; (\rho\cdot\eta)\zeta \;=\; 2(z_3\,\eta \;-\; y_3\zeta)$$

to see that the expression

$${}^{\mathbf{T}}\rho \;\times\; \big(a_3\,\beta\times\gamma \;+\; b_3\,\gamma\times\alpha \;+\; c_3\,\alpha\times\beta\big)$$

is a sum of six terms that cancel each other in pairs.■

As before, the formulation of the dual theorem is left to the reader as an exercise.■

EXERCISES

1. Prove that the conclusion of Theorem 21 is still valid if Γ is completely singular, provided it contains at least two points. [*Hint:* The set of singular points is a geometrical subspace.]

2. Find the equations of the conics in $\mathbb{RP}^2$ which pass through the following five points: with the following homogeneous coordinates:

(i) The five points with the following homogeneous coordinates:

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}1\\-1\\1\end{pmatrix} \quad \begin{pmatrix}1\\-1\\-1\end{pmatrix} \quad \begin{pmatrix}1\\1\\-1\end{pmatrix} \quad \begin{pmatrix}2\\0\\2\end{pmatrix}$$

(ii) The five points with the following homogeneous coordinates:

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \quad \begin{pmatrix}1\\0\\1\end{pmatrix} \quad \begin{pmatrix}0\\1\\1\end{pmatrix} \quad \begin{pmatrix}1\\2\\2\end{pmatrix} \quad \begin{pmatrix}1\\2\\3\end{pmatrix}$$

3. Let Φ be the collineation of $\mathbb{RP}^2$ sending the point with homogeneous coordinates ${}^{\mathbf{T}}(1 \quad 1 \quad 2)$ to ${}^{\mathbf{T}}(2 \quad 2 \quad 1)$, and the lines with homogeneous coordinates

$$(2 \quad 0 \quad -1) \qquad (1 \quad -1 \quad 0) \qquad (1 \quad 1 \quad -1)$$

to the lines with homogeneous coordinates

$$(1 \quad 0 \quad -2) \qquad (1 \quad 1 \quad -4) \qquad (1 \quad 2 \quad -6)$$

respectively. Find the equation of the conic Γ for which Φ is a Steiner collineation (using Theorem 22).

4. Let Γ be the ellipse defined by the affine equation $x^2 + 3y^2 = 4$ in $\mathbb{R}^2$ (hence its projectivization has no ideal points). If T is the unique projective collineation of $\mathbb{RP}^2$ sending $\mathrm{J}(\pm 2, 0)$ and $\mathrm{J}(0, -\frac{2}{3}\sqrt{3})$ to themselves, and sending $\mathrm{J}(-1,1)$ to $\mathrm{J}(1,1)$, then T is a Steiner collineation for Γ. Likewise, if S is the unique projective collineation of $\mathbb{RP}^2$ sending $\mathrm{J}(\pm 2, 0)$ and $\mathrm{J}(0, \frac{2}{3}\sqrt{3})$ to themselves, and sending $\mathrm{J}(-1,1)$ to $\mathrm{J}(1,1)$, then S is also a Steiner collineation for Γ. Show that S and T must be distinct projective collineations. [*Hint:* If $S = T$, then this map fixes the four points on Γ where it meets the x- and y-axes. What does the Fundamental Theorem of Projective Geometry imply about $S = T$ in this case?]

5. State the duals of Theorems 19, 21 and 27–31 (the duality principle implies that these dual results are automatically valid).

6. Prove Theorems 27–30 and their converses.

7. Let Γ be a nonsingular conic in $\mathbb{FP}^2$, and let $\{A, B, C\}$ and $\{A', B', C'\}$ be two disjoint noncollinear sets of points on Γ. Prove that the lines AA', BB' and CC' are concurrent. — A drawing and hints appear on the next page.

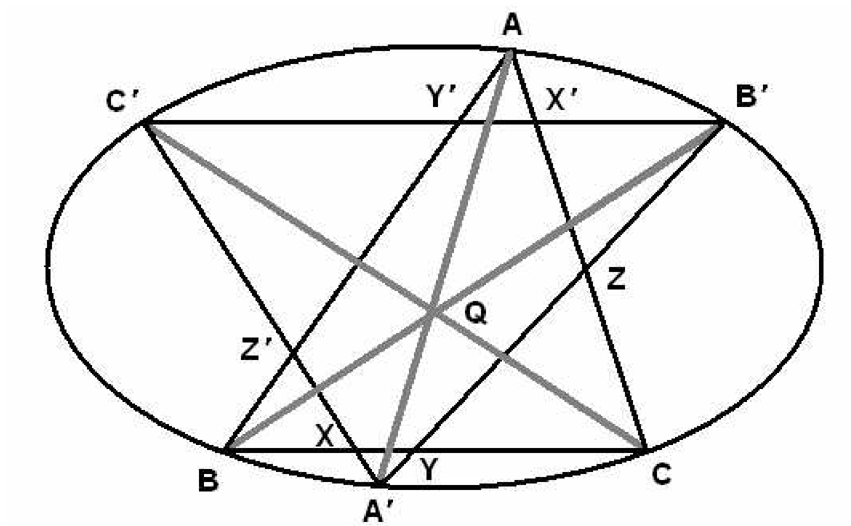

Figure VII.5

[*Hint:* Define X, Y, Z and X', Y', Z' as suggested in the figure. First prove that AA', XX' and YY' are concurrent using Pascal's Theorem. Clearly corresponding results hold for BB' and CC'. Let $U \in C'Y \cap BX'$, and prove that Z', U and Q are collinear by Pappus' Theorem. Also show that ZU, BB' and CC' are concurrent using Pappus' Theorem for $\{C', X', B\}$ and $\{B, Y, C\}$. Finally, apply Pascal's Theorem to $AB'C'A'BC$ to show that BB', CC' and ZZ' are concurrent. Using similar results for XX' and YY' and the previous concurrency relations involving AA', BB' and CC', prove that all six lines are concurrent.]

What is the dual of the preceding result?

8. Let Γ, $\{A, B, C\}$ and $\{A', B', C'\}$ be as in Exercise 6. Prove that the six lines determined by the triangles ABC and $A'B'C'$ form the sides of a hexagon that is tangent to another conic. [*Hint:* The hexagon is $Y'X'ZYXZ'$. Apply the converse to Brianchon's Theorem.]

9. Show that a set of six points on a nonsingular conic Γ determines sixty simple hexagons (in general these sixty hexagons have distinct Pascal lines, and the footnote on page 152 of Fishback contains further information on the totality of all such configurations).