APPENDIX B

THE JOIN IN AFFINE GEOMETRY

In Section II.5 we defined a notion of **join** for geometrical incidence spaces; specifically, if P and Q are geometrical subspaces of an incidence space S, then the join $P \star Q$ is the unique smallest geometrical subspace which contains them both. From an intuitive viewpoint, the name "join" is meant to suggest that $P \star Q$ consists of all points on lines of the form $\mathbf{xy}$, where $\mathbf{x} \in P$ and $\mathbf{y} \in Q$. If S is a projective n-space over some appropriate scalars $\mathbb{F}$, this is shown in Exercise 16 for Section III.4, and the purpose of this Appendix is to prove a similar result for an affine n-space over some $\mathbb{F}$.

Formally, we begin with a generalization of the idea described above.

Definition. Let (S, Π, d) be an abstract geometrical incidence n-space, and let $X \subset S$. Define $\mathbf{J}(X)$ to be the set

$$X \ \cup \ \{ \ \mathbf{y} \in S \mid \mathbf{y} \in \mathbf{uv} \text{ for some } \mathbf{u}, \ \mathbf{v} \in X \ \} \ .$$

Thus $\mathbf{J}(X)$ is X together with all points on lines joining two points of X. Note that the construction of $\mathbf{J}(X)$ from X can be iterated to yield a chain of subsets $X \subset \mathbf{J}(X) \subset \mathbf{J}\big(\mathbf{J}(X)\big) \ \cdots$.

The preceding discussion and definition lead naturally to the following:

QUESTION. *If S is a geometrical incidence n-space and P and Q are geometrical subspaces of S, what is the relationship between $P \star Q$ and $\mathbf{J}(P \cup Q)$? In particular, are they equal, at least if S satisfies some standard additional conditions?*

The exercise from Section III.4 shows that the two sets are equal if S is a standard projective n-space. In general, the next result implies that the two subsets need not be equal. but one is always contained in the other.

Theorem B.1. *In the setting above, we have $\mathbf{J}(P \cup Q) \subset P \star Q$. However, for each $n \geq 2$ there is an example of a* regular *geometrical incidence spaces such that, for some choices of P and Q, the set $\mathbf{J}(P \cup Q)$ is strictly contained in $P \star Q$.*

Proof. The inclusion relationship follows from **G(-2)** and the fact that $P \star Q$ is a geometrical subspace of S. On the other hand, if we take the affine incidence space structure associated to $\mathbb{Z}_2^n$ for $n \geq 2$, then for every subset $X \subset \mathbb{Z}^n$ we automatically have $\mathbf{J}(X) = X$ because every line consists of exactly two points. Thus if W and U are vector subspaces of $\mathbb{Z}_2^n$ such that neither contains the other, then $\mathbf{J}(W \cup U)$ is not a vector subspace. Since $\mathbf{0} \in W \cap U$, we know that $W \star U$ is the vector subspace $W + U$ by Theorem II.36, and it follows in this case that $\mathbf{J}(W \cup U)$ is strictly contained in $W \star U$.■

Note that the examples constructed in the proof are in fact *affine* incidence spaces. The main objective of this appendix is to prove that $\mathbf{J}(P \cup Q) \ = \ P \star Q$ if V is a vector space of dimension ≥ 2 over a field $\mathbb{F}$ which is not (isomorphic to) $\mathbb{Z}_2$.

Theorem B.2. *Let V be a vector space of dimension ≥ 2 over a field $\mathbb{F}$ which is not (isomorphic to) $\mathbb{Z}_2$, and suppose that $P = \mathbf{a} + U$ and $Q = \mathbf{b} + W$ are geometrical subspaces of V. Then the following hold:*

(i) The join $P \star Q$ is the affine span of $P \cup Q$.

(ii) $P \star Q \;=\; \mathbf{J}(P \cup Q)$.

Proof. FIRST STATEMENT. If R is the affine span of P and Q, then R is am affine subspace containing P and Q by Theorem II.19, Theorem II.16 and Exercise 1 for Section II.2 (this is where we use the assumption that $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_2$). Therefore it follows that R also contains $P \star Q$. On the other hand, if R' is a geometrical subspace containing P and Q, then by Theorem II.18 it contains all affine combinations of points in $P \cup Q$, and hence R' must contain R. Combining these observations, we conclude that R must be equal to $P \star Q$.

SECOND STATEMENT. By the previous theorem we know that $\mathbf{J}(P \cup Q) \subset P \star Q$, so it suffices to show that we also have the converse inclusion $P \star Q \subset \mathbf{J}(P \cup Q)$.

Let $\mathbf{x} \in P \star Q$, and let $\{\mathbf{d}_0, \;\cdots\;, \mathbf{d}_p\}$ and $\{\mathbf{c}_0, \;\cdots\;, \mathbf{c}_q\}$ be affine bases for P and Q respectively. Then by the conclusion of the first part of the theorem we may write

$$\mathbf{x} \;=\; \sum_{i=0}^{p} r_i \mathbf{d}_i \;+\; \sum_{j=0}^{p} s_j \mathbf{c}_j$$

where $\sum_i \, r_i \;+\; \sum_j \, s_j \;=\; 1$. Let $t = \sum_i \, r_i$, so that $\sum_j \, s_j \;=\; 1 - t$. There are now two cases, depending upon whether either or neither of the numbers t and $1-t$ is equal to zero. If $t = 0$ or $1 - t = 0$ (hence $t = 1$), then we have $\mathbf{x} \in P \cup Q$. Suppose now that both t and $1-t$ are nonzero. If we set

$$\alpha \;=\; \sum_{i=0}^{p} \frac{r_i}{t} \cdot \mathbf{d}_i \qquad\qquad \beta \;=\; \sum_{j=0}^{q} \frac{s_j}{(1-t)} \cdot \mathbf{c}_j$$

then $\alpha \in P$, $\beta \in Q$, and $\mathbf{x} \;=\; t\,\alpha \;+\; (1-t)\,\beta$; therefore it follows that $\mathbf{x} \in \mathbf{J}(P \cup Q)$.■