APPENDIX C

REVERSAL OF MULTIPLICATION IN SKEW-FIELDS

If (S, Π, d) is a geometrical incidence n-space, then the methods and results of Section IV.4 show that there is a dual geometrical incidence n-space (S, Π, d) is a geometrical incidence whose points are the hyperplanes of S, and such that there is a canonical 1–1 correspondence between the k-planes of S^* and the $(n-k-1)$-planes of S for all k such that $-1 \leq k \leq n$. If $\mathbb{F}$ is a (commutative) field and $S = \mathbb{FP}^n$, then the material on analytic projective geometry in Sections V.1 and VI.1 these notes shows that S^* is also isomorphic to $S = \mathbb{FP}^n$.

Isomorphisms between the projective n-space $\mathbb{FP}^n$ and its dual are called *polarities*, and they play important roles in projective geometry; one class of examples is discussed in Chapter VII of these notes. By the preceding discussion, we know that polarities exist if $\mathbb{F}$ is commutative, and it is natrual to ask if this generalizes:

> *Let $S = \mathbb{FP}^n$ be a projective n-space, where $n \geq 2$ and $\mathbb{F}$ is a skew-field which is not necessarily commutative. Is there an incidence space ieomorphism from S to its dual projective n space $S*$?*

The purpose of this Appendix is to discuss this problem and the related algebraic questions. Unfortunately, the discussion ultimately involves algebraic input that goes far beyond the undergraduate level, so we shall merely sketch the main ideas and give references for additional information on the underlying algebra.

We begin by summarizing a point that is mentioned in Example 2 from Section IV.4 and is discussed further in Sections V.1 and VI.1. If $\mathbb{F}$ is a skew-field and $n \geq 2$ is an integer, then the dual projective n-space $(\mathbb{FP}^n)^*$ associated to $\mathbb{FP}^n$ is isomorphic to $(\mathbb{F}^{\mathsf{OP}})\mathbb{P}^n$, where $\mathbb{F}^{\mathsf{OP}}$ is the skew-field obtained from $\mathbb{F}$ by reversing the order of multiplication in $\mathbb{F}$. Of course, if $\mathbb{F}$ is commutative, then the multiplications in $\mathbb{F}$ and $\mathbb{F}^{\mathsf{OP}}$ are identical, and therefore it follows that S^* is isomorphic to S in such cases. In order to handle the general situation, we must describe the relationship between the two skew-fields $\mathbb{F}$ and $\mathbb{F}^{\mathsf{OP}}$ in the noncommutative case and derive its implications for the relationship between $\mathbb{FP}^n$ and the isomorphic dual objects $(\mathbb{F}^{\mathsf{OP}})\mathbb{P}^n \cong (\mathbb{FP}^n)^*$.

In Appendix A we described a skew-field $\mathbb{K}$ called the *quaterions* which is not commutative, and thus it is natural to ask whether $\mathbb{KP}^n$ and $(\mathbb{K}^{\mathsf{OP}})\mathbb{P}^n \cong (\mathbb{KP}^n)^*$ are isomorphic. By the general classification results, this will happen if and only if there is an algebraic isomorphism between $\mathbb{K}$ and $\mathbb{K}^{\mathsf{OP}}$; note that since $\mathbb{K}$ is not commutative this map cannot be the identity.

In fact, there is an operation on $\mathbb{K}$ called *quaternionic conjugation* which is defined by

$$(a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k})^* = a_1 - a_2\mathbf{i} - a_3\mathbf{j} - a_4\mathbf{k}$$

(see Birkhoff and MacLane, pp. 222–225, especially Exercise 5, p. 224) which has the following properties:

$$\left(\mathbf{x}^*\right)^* = \mathbf{x} \qquad (\mathbf{x} + \mathbf{y})^* = \mathbf{x}^* + \mathbf{y}^* \qquad (\mathbf{x} \cdot \mathbf{y})^* = \mathbf{y}^* \cdot \mathbf{x}^*$$

Note that conjugation, like matrix transposition, reverses the order of multiplication. This conjugation map is 1–1 and onto by the first identity in the display, and the other two identities imply it defines an isomorphism from $\mathbb{K}$ to $\mathbb{K}^{\mathsf{OP}}$.

More generally, by Theorem V.10 we have the following:

Theorem C.1. *Let $\mathbb{F}$ be a skew-field and let $S = \mathbb{F}\mathbb{P}^n$, where $n \geq 2$. Then S is isomorphic to its projective dual space S^* if and only if $\mathbb{F}$ is isomorphic to $\mathbb{F}^{\mathsf{OP}}$.*

Therefore *the existence of projective incidence n-spaces that are not isomorphic to their projective duals* will follow **if** *there are skew-fields $\mathbb{F}$ such that $\mathbb{F}$ is not isomorphic to $\mathbb{F}^{\mathsf{OP}}$.*

Although there are many examples of skew-fields $\mathbb{F}$ for which $\mathbb{F}$ and $\mathbb{F}^{\mathsf{OP}}$ are not isomorphic, describing them is quite nontrivial. The simplest example of this type is given on pages 31–38 of the book by Blanchard listed below; in fact, the exercises on pages 37–38 yield a direct proof that $\mathbb{F}$ and $\mathbb{F}^{\mathsf{OP}}$ are not isomorphic in this case. However, we shall also outline a more comprehensive approach which yields this fact for Blanchard's example and allows one to find systematic infinite families of such skew-fields.

Given a skew-field $\mathbb{L}$, its *center* consists of all $a \in \mathbb{L}$ such that $a \cdot x = x \cdot a$ for all $x \in \mathbb{L}$. This center is always nontrivial, and in particular it contains the smallest subfield containing the unit element (which is isomorphic to either the rational numbers $\mathbb{Q}$ or one of the finite fields $\mathbb{Z}_p$). Furthermore, it is immediate that the center is a commutative subfield that we shall call $\mathbb{F}$, and it also follows that $\mathbb{L}$ is a vector space over $\mathbb{F}$. For example, if $\mathbb{L}$ is the quaternions, then $\mathbb{F}$ is the real numbers. Conversely, given a field $\mathbb{F}$ we can consider all skew-fields $\mathbb{L}$ with center $\mathbb{F}$ such that $\mathbb{L}$ is a *finite-dimensional* vector space over $\mathbb{F}$. Since each such $\mathbb{L}$ is isomorphic to a subring of a finite-dimensional matrix algebra over $\mathbb{F}$ (see Birkhoff and MacLane, pp. 226–227), the isomorphism classes of such objects form a set. A basic construction in algebra makes this set into an abelian group called the **Brauer group** of $\mathbb{F}$ and written $\mathrm{Br}(\mathbb{F})$. The details of the definition appear in the books by Adamson, Herstein, and Jacobson, and also in the papers by Serre (all of which are listed below), and the book by Gille and Szamuely covers the subject thoroughly (but at an extremely high level). The following online references might also be helpful:[1]

`http://en.wikipedia.org/wiki/Division__ring`

`http://en.wikipedia.org/wiki/Field__theory__(mathematics)`

`http://en.wikipedia.org/wiki/Quaternion`

`http://en.wikipedia.org/wiki/Brauer__group`

(Note that there are **two** underscore characters separating words in these links.)

The construction of $\mathrm{Br}(\mathbb{F})$ has the following immediate consequence:

Theorem C.2. *Suppose that the commutative field $\mathbb{F}$ admits no automorphisms other than the identity (for example, this holds if $\mathbb{F}$ is the rational numbers $\mathbb{Q}$ or the real numbers $\mathbb{R}$). Let $\mathbb{L}$ be a skew-field $\mathbb{L}$ which is finite-dimensional over its center $\mathbb{F}$, and assume further that $\mathbb{L}$ is isomorphic to Lop. Then $\mathbb{L}$ determines an element of order 2 in the Brauer group* $\mathrm{Br}(\mathbb{F})$.■

[1]In any citation of Wikipedia articles, it is important to recognize concerns about the accuracy of articles that are submitted by volunteers and subject to editing by a vast number of individuals whose views or understanding may be controversial or unreliable. This issue has been noted explicitly by Wikipedia in its article on itself, where the issue is discussed in some detail. However, for our purposes such questions do not cause problems because I have seen the articles cited below and found them to be accurate.

Thus the proof that $\mathbb{L}$ is not isomorphic to $\mathbb{L}^{\mathsf{OP}}$ for Blanchard's example reduces to showing the following:

(*i*) The center of $\mathbb{L}$ is the rational numbers $\mathbb{Q}$.

(*ii*) The order of $\mathbb{L}$ in the Brauer group is greater than 2.

The first assertion is verified in Blanchard's book, and the second one follows from two easily stated facts:

(*iii*) By construction, the dimension of $\mathbb{L}$ over $\mathbb{Q}$ is equal to 9.

(*iv*) By the results on pages 93–95 of Artin, Nesbitt and Thrall, the order m of $\mathbb{L}$ in $\mathrm{Br}(\mathbb{F})$ satisfies $m^2 = \dim_{\mathbb{Q}} \mathbb{L}$, so that $m = 3$. In particular, $\mathbb{L}$ determines an element of odd order in the Brauer group of $\mathbb{Q}$ and hence cannot be isomorphic to $\mathbb{L}^{\mathsf{OP}}$.

In fact, very powerful methods exist for calculating Brauer groups, and these objects turn out to be fundamentally important to an algebraic subject called *class field theory*. Numerous references are given below. One consequence is that elements of order p exist in $\mathrm{Br}(\mathbb{Q})$ for every prime p (compare Adamson, pp. 220–221).

REFERENCES FOR APPENDIX C

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`http://www.numdam.org/numdam-bin/feuilleter?id=3DSHC_1950-1951__3_`

J.-P. Serre. Local Class Field Theory. *Algebraic Number Theory* (Proceedings of the NATO Advanced Study Institute, 1965), pp. 129–163. Thompson Publishing, Washington, DC, 1967.

[2] In this link, there are **two** underscore characters before the 3.