AFFINE CLASSIFICATION OF HYPERQUADRICS

(Revised and expanded, April 2014)

In Chapter 7 of the notes `http://math.ucr.edu/~res/progeom/pg-all.pdf` there is a complete classification of projective hyperquadrics over the real and complex numbers (see Theorems VII.15 and 16 on page 160). Furthermore, Exercise 4 on page 162 states that every affine real hyperquadric in $\mathbb{R}^n$ is affinely equivalent to an example whose defining equation appears in the following list:

$$(i)\quad x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 = 0 \qquad (1 \le p \le r \le n, r \ge 1, p \ge \tfrac{1}{2}r)$$

$$(ii)\quad x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 + 1 = 0 \qquad (0 \le p \le r \le n, r \ge 1)$$

$$(iii)\quad x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2 + x_{r+1} = 0 \qquad (1 \le p \le r < n, r \ge 1, p \ge \tfrac{1}{2}r)$$

There is a corresponding result over the complex numbers which can be proved by the same methods, but in the complex case the list of examples is simpler:

$$(i)\quad x_1^2 + \cdots + x_r^2 = 0 \qquad (1 \le r \le n)$$

$$(ii)\quad x_1^2 + \cdots + x_r^2 + 1 = 0 \qquad (1 \le r \le n)$$

$$(iii)\quad x_1^2 + \cdots + x_r^2 + x_{r+1} = 0 \qquad (1 \le r < n)$$

It is natural to ask whether these lists are minimal or irredundant in the sense that a nontrivial (in particular, nonempty) quadric is affinely equivalent to **exactly one** of the listed examples, and our goal here is to prove this fact.

Augmented matrices and projectivizations

Recall that an affine hyperquadric Σ in $\mathbb{F}^n$ (where $\mathbb{F}$ can be the real numbers, the complex numbers, or more generally any field in which $1+1 \neq 0$) is definable as the set of all n-dimensional vectors $\mathbf{x}$, viewed as $n \times 1$ column vectors, such that

$$^{\mathbf{T}}\mathbf{x}\, A\, \mathbf{x} \;+\; 2 \cdot {}^{\mathbf{T}}\mathbf{b}\, \mathbf{x} \;+\; c \;=\; 0$$

where A is some nonzero symmetric $n \times n$ matrix and $\mathbf{b}$ is some n-dimensional vector again viewed as a column vector. We may rewrite this more concisely using a single $(n+1) \times (n+1)$ matrix in block form; specifically, the hyperquadric is the set of all $\mathbf{x}$ such that

$$\left({}^{\mathbf{T}}\mathbf{x} \quad 1 \right) \cdot \begin{pmatrix} A & \mathbf{b} \\ {}^{\mathbf{T}}\mathbf{b} & c \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \;=\; 0\,.$$

We shall say that the $(n+1) \times (n+1)$ matrix is an augmented symmetric matrix for the equation of the hyperquadric. This matrix is also a defining matrix for an associated projective hyperquadric Σ' such that $\Sigma = \Sigma' \cap \mathbb{F}^n$.

Definition. Suppose that $\mathbb{F}$ is a field in which $1+1 \neq 0$. A hyperquadric Σ in $\mathbb{F}^n$ is said to be **affinely nondegenerate** if there is a projective hyperquadric Σ' in $\mathbb{FP}^n$ such that $\Sigma = \Sigma' \cap \mathbb{F}^n$ and Σ has at least one nonsingular point.

Since hyperquadrics and nonsingularity are preserved by projective collineations, it follows that two affinely equivalent hyperquadrices are either both nondegenerate or both not nondegenerate (*i.e.*, they are *degenerate*).

One motivation for this definition is Theorem VII.6 on page 151 of `pgnotes07.pdf`, and by Theorem E.5 on page 210 of

`http://math.ucr.edu/~res/progeom/pgnotesappe.pdf`

all complex hyperquadrics are nondegenerate. The real case is more complicated, but there are fairly simple characterizations of nondegenerate real hyperquadrics:

PROPOSITION. *Suppose that Σ is a real hyperquadric in $\mathbb{R}^n$ which has a defining equation corresponding to the following augmented symmetric matrix (in which A is symmetric):*

$$\begin{pmatrix} A & \mathbf{b} \\ {}^{\mathbf{T}}\mathbf{b} & c \end{pmatrix}$$

Then Σ is nondegenerate if either of the following holds:

(i) *The augmented symmetric matrix has both positive and negative eigenvalues.*

(ii) *Σ is not an affine k-plane for some $k \geq 0$.*

Proof. We shall go down the list to see which of the standard examples extends to a projective hyperquadric with at least one singular point. Here are the homogeneous coordinate equations for the naturally associated projective hyperquadrics in each case:

$$(i)\quad u_1^2 + \cdots + u_p^2 - u_{p+1}^2 - \cdots - u_r^2 = 0 \qquad (1 \leq p \leq r \leq n, r \geq 1, p \geq \tfrac{1}{2}r)$$

$$(ii)\quad u_0^2 + u_1^2 + \cdots + u_p^2 - u_{p+1}^2 - \cdots - u_r^2 = 0 \qquad (0 \leq p \leq r \leq n, r \geq 1)$$

$$(iii)\quad u_1^2 + \cdots + u_p^2 - u_{p+1}^2 - \cdots - u_r^2 + u_0 u_{r+1} = 0 \qquad (1 \leq p \leq r < n, r \geq 1, p \geq \tfrac{1}{2}r)$$

Suppose that $p < r$ in each case. In the first and third cases, consider the point X whose coordinates are given by $u_p = u_{p+1} = 1$ and all other coordinates equal to zero; then X is a nonsingular point of Σ. In the second case, consider the point Y whose coordinates are given by $u_p = 1, u_{p+1} = \sqrt{2}$ and all other coordinates equal to zero; then Y is a nonsingular point of Σ. Therefore the standard examples are nondegenerate except possibly in cases where $p = r$.

In the third case where $p = r$, consider the point Z whose coordinates are given by $u_r = 1, u_{r+1} = -1$ and all other coordinates equal to zero; then Z is a nonsingular point of Σ. Thus the only possible degenerate examples are those defined by homogeneous quadratic polynomials of the form $\sum_i u_i^2 = 0$, where the sum runs over some subset of $\{0, 1, \cdots, n\}$. Conversely, a nonempty projective hyperquadric defined by such an equation has the property that every point is a singular point (in fact, the set of points satisfying the equation is the set of all points satisfying the system of linear homogeneous equations of the form $u_i = 0$, where i runs through all indices in the summation.

To prove (i), notice that the augmented matrix has both positive and negative eigenvalues, then the naturally associated projective quadric is not given by an equation of the form $\sum_i u_i^2 = 0$, and therefore this projective hyperquadric has a nonsingular point. To prove (ii), observe that if the original hyperquadric is degenerate, then by the preceding paragraph it is the intersection of a projective k-plane with $\mathbb{F}^n$ and hence must be an affine k-plane.∎

Equations defining the same affine hyperquadric

In Section VII.2 of `pgnotes07.pdf`, one result (Theorem VII.6) shows that two projective hyperquadrics which satisfy a very weak nondegeneracy hypothesis will be equivalent if and only if the defining equation for one is a nonzero multiple of the defining equation for the other, at least if $1 + 1 \neq 0$ in $\mathbb{F}$. We are going to need a similar result for affine hyperquadrics. In order to keep the discussion as elementary and concise as possible, we shall restrict attention to the case where $\mathbb{F}$ is the field of real numbers. The approach will be analogous to that in Theorem VII.6 from `pgnotes07.pdf`, but there are significant complications. The reason for this is that lines through a nonsingular point in a projective quadric are easily classified as tangent or secant lines depending upon their intersections with the quadric, but the corresponding classification in the affine case includes an additional possibility; namely, there might be some lines which are not tangents but meet the affine quadric in exactly one point. — For example, the latter happens if we consider the intersection of the hyperbola with equation $xy - 1 = 0$ and a vertical line with equation $x = a$ (where $a \neq 0$). In the ordinary coordinate plane, the only intersection point has coordinates (a, a^{-1}), if we extend the curves to the projective plane we obtain a second intersection point; namely, the point at infinity on the vertical line (note that the points at infinity on the projectivized hyperbola are the ideal points on the $x-$ and $y-$axes).

THEOREM ON DEFINING EQUATIONS. *Let Σ an the affinely nondegenerate hyperquadric in $\mathbb{R}^n$ defined by each of the augmented symmetric matrices*

$$Q_i \quad = \quad \begin{pmatrix} A_i & \mathbf{b}_i \\ {}^{\mathbf{T}}\mathbf{b}_i & c_i \end{pmatrix}, \qquad (i \quad = \quad 1, 2)$$

where each A_i is a symmetric matrix, each $\mathbf{b}_i$ is a column vector and each c_i is a scalar. Then there is a nonzero constant k such that $Q_1 = k\, Q_2$.

The preceding theorem implies a strong result on the standard way of passing from projective to affine hyperquadrics. Recall that if $\Sigma \subset \mathbb{RP}^n$ is a projective hyperquadric defined by the homogeneous polynomial $f(x_1, \cdots, x_n, x_{n+1})$ such that the monomial x_{n+1} does not divide f, then an associated affine hyperquadric $A = \Sigma \cap \mathbb{R}^n$ is defined by the inhomogeneous quadratic polynomial $f(x_1, \cdots, x_n, 1) = 0$.

COROLLARY. *Let Σ_1 and Σ_2 be hyperquadrics in $\mathbb{RP}^n$ such that the following hold:*

(i) *Neither Σ_1 nor Σ_2 has a defining equation which is divisible by x_{n+1}.*

(ii) *Both Σ_1 and Σ_2 define the same affine hyperquadric A.*

(iii) *The affine hyperquadric A is affinely nondegenerate.*

Then $\Sigma_1 = \Sigma_2$.

Proof of the corollary. Let Q_1 and Q_2 be symmetric matrices which define Σ_1 and Σ_2, and let A_i be the affine hyperquadric associated to Σ_i, so that $A_1 = A_2$. The hypotheses guarantee that this affine hyperquadric is affinely nondegenerate, and therefore by the theorem we know that $Q_1 = k\, Q_2$ for some nonzero constant k. As noted before, this means that $\Sigma_1 = \Sigma_2$ because they have defining equations which are nonzero multiples of each other.■

In the statement of the theorem, note that if a point $\mathbf{x}$ on Σ is nonsingular with respect to one defining equation, then the results of `pgnotesappe.pdf` show that it is nonsingular with respect to every other defining equation, for in this case the tangent space at $\mathbf{x}$ is a hyperplane which consists of all points on lines through $\mathbf{x}$ which are tangent lines to differentiable curves in Σ which pass through $\mathbf{x}$.

Proof of the Theorem on Defining Equations. Let $\mathbf{x}$ be a nonsingular point of Σ. Then the observations in the preceding paragraph imply that one obtains the same tangent hyperplane at $\mathbf{x}$ from each of the two defining equations. If we let

$$\xi \quad = \quad \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$$

this means that homogeneous coordinates for the tangent hyperplane are given by ${}^{\mathbf{T}}\xi\, Q_1$ and ${}^{\mathbf{T}}\xi\, Q_2$, which implies that there is some nonzero contant k such that

$${}^{\mathbf{T}}\xi\, Q_1 \quad = \quad k \cdot {}^{\mathbf{T}}\xi\, Q_2 \ .$$

Another property of Σ which does not depend upon the choice of defining equation involves secant lines though $\mathbf{x}$ which meet Σ at some second point in $\mathbb{R}^n$: If $\mathbf{v}$ is a nonzero vector such that the line through $\mathbf{x}$ with direction $\mathbf{v}$ also meets Σ at $\mathbf{x} + s^*\,\mathbf{v}$ for some unique $s^* \neq 0$, this yields an equation involving Q_1 and a similar equation involving Q_2. Specifically, if

$$\theta \quad = \quad \begin{pmatrix} \mathbf{v} \\ 0 \end{pmatrix}$$

then the two roots of the quadratic equations

$$0 \quad = \quad \left({}^{\mathbf{T}}\xi \ + \ s\,{}^{\mathbf{T}}\theta\right) Q_i \ (\xi \ + \ s\,\theta) \quad = \quad 2 \cdot s\,{}^{\mathbf{T}}\xi\, Q_i\, \theta \ + \ s^2\,{}^{\mathbf{T}}\theta\, Q_i\, \theta \qquad (i \ = \ 1,\ 2)$$

are $s = 0$ and $s = s^*$. Since there are only two roots, the coefficients of s^2 in these equations must be nonzero, and likewise for the coefficients of s. Furthermore, since ${}^{\mathbf{T}}\xi\, Q_1 = k\,{}^{\mathbf{T}}\xi\, Q_2$ we must have

$$s^* \quad = \quad -\,\frac{2\,s\,{}^{\mathbf{T}}\xi\, Q_i\, \theta}{{}^{\mathbf{T}}\theta\, Q_i\, \theta} \qquad (i \ = \ 1,\ 2)$$

and $s^* \neq 0$ the denominators of these fractions must both be nonzero. These observations and some elementary rewriting or equations imply that

$${}^{\mathbf{T}}\theta\, Q_1\, \theta \quad = \quad {}^{\mathbf{T}}\theta\, Q_2\, \theta$$

for all $\theta = \left({}^{\mathbf{R}}\mathbf{v} \quad 0\right)$ satisfying the conditions in the preceding paragraph. By construction we may rewrite the displayed equation in the form

$${}^{\mathbf{T}}\mathbf{v}\, A_1\, \mathbf{v} \quad = \quad {}^{\mathbf{T}}\mathbf{v}\, A_2\, \mathbf{v}$$

where A_i is given as in the statement of the theorem.

To be more precise, our reasoning shows that the last displayed equation is valid provided the line through $\mathbf{x}$ with direction $\mathbf{v}$ is neither a tangent line to Σ at $\mathbf{x}$ nor a line whose point at infinity lies on one or more of the projective hyperquadrics quadrics $\mathbb{P}(Q_i)$ defined by Q_i. We shall show that the set of good choices for $\mathbf{v}$ is the defined by the nonzero sets of three nontrivial polynomials; in other words, the displayed formula is valid for $\mathbf{v}$ provided $g_i(\mathbf{v}) \neq 0$ for $1 \leq i \leq 3$, where each g_i

is a nontrivial polynomial in the coordinates of $\mathbf{v}$. — A normal vector to the tangent hyperplane at $\mathbf{x}$ is given by

$$A_1\mathbf{x} \ + \ \mathbf{b}_1 \quad = \quad k\ (A_2\mathbf{x} \ + \ \mathbf{b}_2)$$

so the condition for $\mathbf{v}$ to be a tangential direction at $\mathbf{x}$ is given by a first degree polynomial in the coordinates of $\mathbf{v}$. Next, the point at infinity for the line through $\mathbf{x}$ with direction $\mathbf{v}$ has homogeneous coorindates θ, so the defining equation for the points at infinity on $\mathbb{P}(Q_i)$ is $0 \ = \ {}^{\mathbf{T}}\theta\, Q_i\, \theta$, and since the last coordinate of θ is zero these reduce to $0 \ = \ {}^{\mathbf{T}}\mathbf{v}\, A_i\, \mathbf{v}$ for $i = 1, 2$. To recapitulate, if $\mathbf{v}$ does not satisfy any of these equations then we have

$${}^{\mathbf{T}}\mathbf{v}\, A_1\, \mathbf{v} \quad = \quad k \cdot {}^{\mathbf{T}}\mathbf{v}\, A_2\, \mathbf{v}\ .$$

By the Sparseness Theorem and its corollaries (see the Addendum), the set of all $\mathbf{v}$ not satisfying the three given polynomial equations is an open dense subset of $\mathbb{R}^n$. Since two continuous real valued functions on $\mathbb{R}^n$ are equal if they have the same values on a dense subset, it follows that the displayed equation holds for **ALL** choices of $\mathbf{v}$ in $\mathbb{R}^n$. If we change notation from $\mathbf{v}$ to θ, then we may rewrite the identity of the preceding sentence in the form

$${}^{\mathbf{T}}\theta\, Q_1\, \theta \quad = \quad k \cdot {}^{\mathbf{T}}\theta\, Q_2\, \theta$$

where $\theta \in \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. If we now use the identities

$$2 \cdot {}^{\mathbf{T}}\theta_1\, Q_i\, \theta_2 \quad = \quad {}^{\mathbf{T}}(\theta_1 + \theta_2)\, Q_i\, (\theta_1 + \theta_2) \ - \ {}^{\mathbf{T}}\theta_1\, Q_i\, \theta_1 \ - \ {}^{\mathbf{T}}\theta_2\, Q_i\, \theta_2$$

we see that ${}^{\mathbf{T}}\theta_1\, Q_1\, \theta_2 = k \cdot {}^{\mathbf{T}}\theta_1\, Q_2\, \theta_2$ for all $\theta_1, \theta_2 \in \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$.

We now claim that a similar identity holds if η_1 and η_2 are arbitrary vectors in $\mathbb{R}^{n+1}$. Since the last coordinate of ξ is 1, we can write an arbitrary vector $\eta_j \in \mathbb{R}^{n+1}$ as a sum $\theta_j + z_j\, \xi$ where the last coordinate of θ_j is zero and $z_j \in \mathbb{R}$. We then have

$${}^{\mathbf{T}}\eta_1\, Q_1\, \eta_2 \quad = \quad {}^{\mathbf{T}}\theta_1\, Q_1\, \theta_2 \ + \ z_1\, {}^{\mathbf{T}}\xi\, Q_1\, \theta_1 \ + \ z_2\, {}^{\mathbf{T}}\xi\, Q_1\, \theta_2 \ + \ {}^{\mathbf{T}}\xi\, Q_1\, \xi\ .$$

By our previous discussions in this proof the first term is $k\, {}^{\mathbf{T}}\theta_1\, Q_2\, \theta_2$, and the equation ${}^{\mathbf{T}}\xi\, Q_1 = k\, {}^{\mathbf{T}}\xi\, Q_2$ implies the equalities

$$z_1\, {}^{\mathbf{T}}\xi\, Q_1\, \theta_1 \quad = \quad k\, z_1\, {}^{\mathbf{T}}\xi\, Q_2\, \theta_1\ , \quad z_2\, {}^{\mathbf{T}}\xi\, Q_1\, \theta_2 \quad = \quad k\, z_2\, {}^{\mathbf{T}}\xi\, Q_1\, \theta_2\ , \quad {}^{\mathbf{T}}\xi\, Q_1\, \xi \quad = \quad k\, {}^{\mathbf{T}}\xi\, Q_1\, \xi$$

and if we combine these we find that ${}^{\mathbf{T}}\eta_1\, Q_1\, \eta_2 = k \cdot {}^{\mathbf{T}}\eta_1\, Q_2\, \eta_2$ for all $\eta_1, \eta_2 \in \mathbb{R}^{n+1}$.

Finally, if we take η_1 and η_2 to be the standard unit vectors ε_u and ε_v in $\mathbb{R}^{n+1}$ then ${}^{\mathbf{T}}\eta_1\, Q_i\, \eta_2$ is the (u, v) entry of Q_i, and consequently we see that for all u and v the (u, v) entry of Q_1 is k times the corresponding entry for Q_2, so that $Q_1 = k\, Q_2$, which is what we wanted to prove.■

Note. There is an alternate approach to this type of result in Appendix D of the following book:

A. Reventós Tarrida. *Affine Maps, Euclidean Motions and Quadrics*. Springer-Verlag, New York *etc.*, 2011.

A matrix criterion for affine equivalence

The following straightforward reformulation of affine equivalence will be extremely useful.

THEOREM. *Let $\mathbb{R}$ be the of real numbers, and let Σ_i be the nonempty affinely nondegenerate hyperquadric in $\mathbb{R}^n$ defined by the augmented symmetric matrix*

$$\begin{pmatrix} A_i & \mathbf{b}_i \\ {}^{\mathbf{T}}\mathbf{b}_i & c_i \end{pmatrix} , \qquad i \;=\; 1,2$$

where A_i is a symmetric matrix, $\mathbf{b}_i$ is a column vector and $c_i =$ is a scalar. Suppose further that there is an affine transformation $\mathbf{T}(\mathbf{x}) = P\mathbf{x} + \mathbf{q}$ which maps Σ_2 to Σ_1. Then the augmented symmetric matrices are related by the following equation, in which k is some nonzero constant:

$$\begin{pmatrix} {}^{\mathbf{T}}P & 0 \\ {}^{\mathbf{T}}\mathbf{b} & 1 \end{pmatrix} \cdot \begin{pmatrix} A_2 & \mathbf{b}_2 \\ {}^{\mathbf{T}}\mathbf{b}_2 & c_2 \end{pmatrix} \cdot \begin{pmatrix} P & \mathbf{q} \\ 0 & 1 \end{pmatrix} \;=\; k \cdot \begin{pmatrix} A_1 & \mathbf{b}_1 \\ {}^{\mathbf{T}}\mathbf{b}_1 & c_1 \end{pmatrix}$$

Proof. We know that $\mathbf{x}$ lies in Σ_1 if and only if $\mathbf{T}(\mathbf{x})$ lies in Σ_2. The first equation implies that Σ_1 is defined by the augmented symmetric matrix on the right hand side of the displayed formula, while the second implies that Σ_1 is defined by the augmented symmetric matrix on the left hand side of the displayed formula without the constant factor k. Since Σ_2 and Σ_1 are nondegenerate, we can now apply the previous theorem to conclude that each of these augmented symmetric matrices must be a nonzero scalar multiple of the other.■

If we expand the left hand side of the equation in the preceding theorem, we see that it is given by the following expression:

$$\begin{pmatrix} {}^{\mathbf{T}}P\, A_2\, P & {}^{\mathbf{T}}P\,(A_2\,\mathbf{q} + \mathbf{b}_2) \\ \left({}^{\mathbf{T}}\mathbf{q}\, A_2 + {}^{\mathbf{T}}\mathbf{b}_2\right)\, P & {}^{\mathbf{T}}\mathbf{x}\, A_2\, \mathbf{x} + 2 \cdot {}^{\mathbf{T}}\mathbf{b}_2\, \mathbf{x} + c_2 \end{pmatrix}$$

We shall need this formula to complete the proof of the classification theorem.

The definiteness index

If Σ is a hyperquadric in real projective n-space which is defined by the symmetric matrix A, then the classificiation for projective quadrics implies that the projective equivalence class of Σ is completely determined by the rank of A and the absolute value of its signature. For the classification of affine hyperquadrics up to affine equivalence, we shall need an invariant which is equivalent to the absolute value of the signature but is the same for a symmetric matrix and its negative (recall that in the projective case A and $-A$ define the same hyperquadric, and clearly an analogous statement holds for affine hyperquadrics).

Definition. Given a symmetric $m \times m$ matrix A over the real numbers, the *definiteness index* $\Delta(A)$ is the maximum dimension of all vector subspaces $V \subset \mathbb{R}^n$ such that the bilinear form associated to A is either positive or negative definite on V. Equivalently, by the diagonalization theorem for real symmetric matrices this is the maxiumum of the dimensions of V_+ and V_-, where V_+ is spanned by the eigenvectors for positive eigenvalues and V_- is spanned by the eigenvectors for negative eigenvalues.

It follows immediately that the definiteness index satsifies $\Delta(A) = \Delta(k\,A)$ for all $k \neq 0$ and that if P is an invertible $m \times m$ matrix then we have

$$\Delta\left({}^{\mathbf{T}}P\, A\, P\right) \;=\; \Delta(A) \;.$$

There is a very simple formula relating the definiteness index to the absolute value of the signature.

PROPOSITION. *Let A be a symmetric $m \times m$ matrix with rank r, and let σ and Δ denote its signature and definiteness index respectively. Then $|\sigma| = 2\,\Delta - r$.*

Proof. In the notation of the paragraph which defines Δ, let p and n be the dimensions of V_+ and V_- respectively. Then we have

$$\Delta \;=\; \max(p,n)\;, \quad r \;=\; p\,+\,n\;, \quad |\sigma| \;=\; |p\,-\,n|$$

and the verification splits into two cases depending upon whether $p \geq n$ or vice versa. If $p \geq n$ then $\Delta = p$, so that $n = r - p$ implies

$$|\sigma| \;=\; p\,-\,n \;=\; p\,-\,(r-p) \;=\; 2p\,-\,r \;=\; 2\Delta\,-\,r$$

and similarly if $n \geq p$ then $\Delta = n$ and we have

$$|\sigma| \;=\; n\,-\,p \;=\; n\,-\,(r-n) \;=\; 2n\,-\,r \;=\; 2\Delta\,-\,r$$

which is the same formula derived in the first case.■

The affine classification

Here is a formal statement of the main result:

THEOREM. *Let Σ be a nondegenerate affine hyperquadric in $\mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then Σ is affinely equivalent to exactly one of the hyperquadrics listed at the beginning of this document.*

Proof. We need to check that two hyperquadrics defined by two different equations in the lists for $\mathbb{R}$ and $\mathbb{C}$ are not affinely equivalent. The argument has two main steps. First, we shall check this is true if both equations are in one of the sublists $(i) - (iii)$ described above. Next, we shall show that the statement is true if the two equations come from different sublists. There are two cases depending upon the field under consideration. Suppose that the augmented matrices for the different equations are given by

$$Q_j \;=\; \begin{pmatrix} A_j & \mathbf{b}_j \\ {}^{\mathbf{T}}\mathbf{b}_j & c_j \end{pmatrix}$$

where $j = 1$ or 2, and we shall let Σ_j be the affine hyperquadric defined by Q_j.

The complex case. If Σ_1 and Σ_2 are affinely equivalent, then the preceding results imply that the ranks of the augmented matrices, which we shall write $\rho(Q_1)$ and $\rho(Q_2)$ are equal, and also (by the previous explicit formula) the ranks $\rho(A_1)$ and $\rho(A_2)$ are equal. If we let r_j be the appropriate number in the defining equation for Σ_j, then we have the following formulas:

(1.1) For an equation of type (i), we have $\rho(Q_j) = \rho(A_j) = r_j$.

(1.2) For an equation of type (ii), we have $\rho(Q_j) = r_j + 1$ and $\rho(A_j) = r_j$.

(1.3) For an equation of type (iii), we have $\rho(Q_j) = r_j + 1$ and $\rho(A_j) = r_j - 1$.

In fact, these hold for both $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$.

If Σ_1 and Σ_2 are affinely equivalent, then as noted above we have $\rho(Q_1) = \rho(Q_2)$ and if they are both defined by an equation of the same type, it follows from (1.1) – (1.3) that $r_1 = r_2$, completing the proof for such cases.

Again by (1.1) – (1.3), if Σ_1 and Σ_2 are affinely equivalent, then we also have

$$\rho(Q_1) \;-\; \rho(A_1) \;=\; \rho(Q_2) \;-\; \rho(A_2)\ .$$

Since this difference is $0, 1, 2$ for equations of types (i), (ii) and (iii) respectively, we see that two equations of different types cannot define affinely equivalent hyperquadrics. This completes the argument in the complex case.

The real case. The arguments in the complex case imply that Σ_1 and Σ_2 are both defined by equations of the same type and that $r_1 = r_2$; let r denote this common value. In particular, it follows that the defining equations for both standard models have the same type (i), (ii) or (iii). Therefore it will suffice to prove that $p_1 = p_2$ must also hold.

In analogy with the complex case, if Σ_1 and Σ_2 are affinely equivalent then the definiteness indices for the defining equations satisfy $\Delta(Q_1) = \Delta(Q_2)$ and $\Delta(A_1) = \Delta(A_2)$. There are now essentially two cases depending upon whether the defining equations for the hyperquadrics have type (i), (ii) or (iii); the phrase "essentially two cases" means that the arguments for types (i) and (iii) are identical.

If Σ_1 and Σ_2 are both defined by equations of type (i) or (iii), then the conditions $p_i \geq \frac{1}{2}\, r_i$ imply that $p_i = \Delta(A_i)$, and hence if Σ_1 and Σ_2 are affinely equivalent then $p_1 = \Delta(A_1) = \Delta(A_2) = p_2$.

Now suppose that Σ_1 and Σ_2 are both defined by equations of type (ii) and that these hyperquadrics are affinely equivalent.

Subcase (a). Suppose that $p_1, p_2 \geq \frac{1}{2}\, r$. Then the argument for types (i) and (iii) implies that $p_1 = \Delta(A_1) = \Delta(A_2) = p_2$.

Subcase (b). Suppose that $p_1, p_2 \leq \frac{1}{2}\, r$. In this subcase we have $n - p_1 = \Delta(A_1) = \Delta(A_2) = n - p_2$, which again yields $p_1 = p_2$.

Subcase (c). If neither of the preceding holds, then either $p_1 > \frac{1}{2}\, r > p_2$ or else $p_2 > \frac{1}{2}\, r > p_1$. *Claim:* In this case we shall prove that the existence of an affinely equivalent pair Σ_1 and Σ_2 leads to a contradiction. — Suppose that such a pair exists. Without loss of generality, we might as well assume that the latter holds. By our hypotheses, the definiteness indices of the various defining matrices for the Σ_i and their projective extensions must satisfy

$$p_1 \;=\; \Delta(A_1) \;=\; \Delta(A_2) \;=\; r \;-\; p_2\ , \qquad p_1 \;+\; 1 \;=\; \Delta(Q_1) \;=\; \Delta(Q_2) \;=\; r \;-\; p_2$$

and this yields a contradiction because $\Delta(Q_1) - \Delta(A_1) = 1$ and $\Delta(Q_2) - \Delta(A_2) = 0$. The source of the contradiction was the assumption that two distinct equations in the type (ii) list (with $r_1 = r = r_2$ but $p_1 \neq p_2$) defined affinely equivalent pairs, and hence it follows that such a pair cannot exist with $p_1 \neq p_2$.■

Example. It is very easy to describe two affinely inequivalent quadric surfaces in $\mathbb{R}^3$ such that one is sent to another by a *nonlinear* change of variables transformation

$$(u, v, w) \;=\; \mathbf{F}(x, y, z)$$

where $\mathbf{F}$ is 1–1 onto, its coordinate functions have partial derivatives of all orders, and the Jacobian of $\mathbf{F}$ is nonzero (so that $\mathbf{F}$ is an invertible mapping). Specifically, if we let $u = x/\sqrt{1+z^2}$, $v = y/\sqrt{1+z^2}$, and $w = z$, then $\mathbf{F}$ maps the one-sheeted hyperboloid with equation $x^2+y^2-z^2 = 1$ to the circular cylinder with equation $u^2 + v^2 = 1$.

Classification up to congruence or similarity

The classification results for affine and projective hyperquadrics over $\mathbb{R}$ and $\mathbb{C}$ lead naturally to related questions in separate directions:

TOPOLOGICAL CLASSIFICATION PROBLEMS. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Given two projective or affine hyperquadrics Σ_1 and Σ_2 in $\mathbb{FP}^n$ or $\mathbb{F}^n$, is there a homeomorphism (= topological equivalence) $\mathbf{T}$ mapping Σ_1 onto Σ_2?*

Actually, there are two possible versions of this question, depending upon whether one is looking for a homeomorphism from $\mathbb{FP}^n$ or $\mathbb{F}^n$ to itself or merely for a homeomorphism from Σ_1 to Σ_2. In this document we shall be interested mainly in the less restrictive (second) option.

CONGRUENCE OR ISOMETRY CLASSIFICATION PROBLEM. *Given two affine hyperquadrics Σ_1 and Σ_2 in $\mathbb{R}^n$, is there a congruence or isometry $\mathbf{T}$ from $\mathbb{R}^n$ to itself mapping Σ_1 onto Σ_2?*

Once again, there are two possible versions of the question. An arbitrary isometry $\mathbf{T}$ from $\mathbb{R}^n$ to itself is an affine transformation of the form $\mathbf{T}(\mathbf{x}) = P\,\mathbf{x} + \mathbf{q}$, where P is some orthogonal matrix (*i.e*, ${}^{\mathbf{T}}P = P^{-1}$ and $\mathbf{q}$ is some vector); in some writings, the term "congruence" refers to an arbitrary isometry, while in others it refers to an isometry for which $\det P = +1$. In this document we shall be interested mainly in the less restrictive (first) option.

We also have a third problem which is closely related to the second:

SIMILARITY CLASSIFICATION PROBLEM. *Given two affine hyperquadrics Σ_1 and Σ_2 in $\mathbb{R}^n$, is there a similarity transformation $\mathbf{T}$ from $\mathbb{R}^n$ to itself mapping Σ_1 onto Σ_2?*

A similarity transformation $\mathbf{T}$ from $\mathbb{R}^n$ to itself is an affine transformation of the form $\mathbf{T}(\mathbf{x}) = P\,\mathbf{x} + \mathbf{q}$, where P is is a positive multiple of some orthogonal matrix (the *ratio of similitude*) and $\mathbf{q}$ is some vector. Note that an isometry is a similarity transformation for which the ratio of similitude is $+1$.

An affine transformation of $\mathbb{F}^n$ (where $\mathbb{F}$ is any field) always has an inverse which is an affine transformation, and if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ then an affine transformation is continuous, and therefore every affine transformation is a homeomorphism. Similarly, if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ then a projective transformation of $\mathbb{FP}^n$ to itself is also a homeomorphism. Using these facts, we can summarize the relationships among the various classification problems as follows:

Congruent or isometric affine hyperquadrics are similar.

Similar affine hyperquadrics are affinely equivalent.

Affinely or projectively equivalent hyperquadrics are topologically equivalent.

At the end of the previous section we gave examples of affinely inequivalent quadric surfaces in $\mathbb{R}^3$ which are topologically equivalent. However, a complete discussion of the topological classification requires concepts from graduate level topology courses, and for this reason the topological classification is discussed in a separate document titled `quadrics2.pdf`. In contrast, the congruence and siliarity classifications only require input from a second course in linear algebra, and these classifications are derived in `quadrics3.pdf`.

ADDENDUM: Sets of solutions to polynomials

If $p(t)$ is a nonzero polynomial with coefficients in a field $\mathbb{F}$, then the set of solutions is a finite subset of $\mathbb{F}$, and for simple examples of real polynomials in two indeterminates one can check directly that the sets of solutions are relatively sparse subsets of $\mathbb{R}^2$. More precisely, if $p(t_1, t_2)$ is a nonzero polynomial with real coefficients, then for each point $(a, b) \in \mathbb{R}^2$ and each $\varepsilon > 0$ there is a point (c, d) such that the distance from (a, b) to (c, d) is less than ε and $p(c, d) \neq 0$. We shall formulate and prove a general version of this result.

Recall (say, from multivariable calculus) that a subset $U \subset \mathbb{R}^n$ is open if for each $\mathbf{x} \in U$ there is some $\delta > 0$ such that $|\mathbf{y} - \mathbf{x}| < \delta$ implies $\mathbf{y} \in U$. Since polynomial functions are continuous, the set of all points $\mathbf{x} \in \mathbb{R}^n$ such that $p(\mathbf{x}) \neq 0$ is open (if p is nonzero at $\mathbf{x}$, it is also nonzero near $\mathbf{x}$). We also need the following concept:

Definition. A subset $A \subset \mathbb{R}^n$ is *dense* in $\mathbb{R}^n$ if for each $\mathbf{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ there is some $\mathbf{a} \in A$ such that $|\mathbf{a} - \mathbf{x}| < \varepsilon$.

One of the simplest, and most important, nontrivial examples of a dense subset is the subset $A = \mathbb{Q}$ of all rational numbers in $\mathbb{R} = \mathbb{R}^1$.

SPARSENESS THEOREM. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.*

(i) If $p(t_1, \cdots, t_n)$ is a polynomial form in $\mathbb{F}[t_1, \cdots, t_n]$, then $V(p)$ is a closed subset of $\mathbb{F}^n$ with respect to the metric topology. Furthermore, if $V(p)$ contains a nonempty open subset, then p is the zero polynomial.

(ii) If $p(t_1, \cdots, t_n)$ is a nonzero polynomial form in $\mathbb{F}[t_1, \cdots, t_n]$, then the complement $\mathbb{F}^n - V(p)$ of $V(p)$ is an open dense subset of $\mathbb{F}^n$.

Similar conclusions hold for the zero sets of a finite family of nonzero polynomials.

COROLLARY. *If $p_1, \cdots, p_m$ are nonzero polynomial forms in $\mathbb{F}[t_1, \cdots, t_n]$, then the complement $\mathbb{F}^n - \cup_j V(p_j)$ of $\cup_j V(p_j)$ is an open dense subset of $\mathbb{F}^n$.*

Proof of the corollary. The conclusion will follow if we know that a finite intersection of open dense subsets is dense, and by an induction argument the latter reduces to proving the result for an intersection of two such subsets. Suppose that U and V are open and dense subsets of $\mathbb{F}^n$, let $\mathbf{x} \in \mathbb{F}^n$, and let $\varepsilon > 0$. Since U is dense, there is some $\mathbf{u} \in U$ such that $|\mathbf{u} - \mathbf{x}| < \frac{1}{2}\varepsilon$. Let $\delta > 0$ such that $\delta < \frac{1}{2}\varepsilon$ and $|\mathbf{y} - \mathbf{u}| < \delta$ implies that $\mathbf{y} \in U$. Then there is some $\mathbf{v} \in V$ such that $|\mathbf{v} - \mathbf{u}| < \delta$. By the choice of δ we know that $\mathbf{v} \in U$, so that $\mathbf{v} \in U \cap V$, and by the Triangle Inequality we know that $|\mathbf{v} - \mathbf{x}| < \varepsilon$.■

Proof of the Sparseness Theorem. (i) The first conclusion follows because p is continuous, so that the inverse image $V(p)$ of the closed set $\{0\} \subset \mathbb{F}^n$ is a closed set.

We shall prove the second conclusion by induction on the number of indeterminates in the polynomial. The result is for polynomials in one indeterminate because a nontrivial polynomial in one variable has only finitely many roots. Assume it is true for polynomials of with $n - 1$ indeterminates, where $n \geq 2$, and write the polynomial in the form

$$p(t_1, \cdots, t_n) \quad = \quad \sum_{j=0}^{d} q_j(t_1, \cdots, t_{n-1})\, t_n^j$$

where we might as well assume that $d > 0$ (otherwise we have a polynomial not involving the indeterminate t_n and the conclusion of the proposition follows from the induction hypothesis).

Suppose now that $p = 0$ on some open subset U, let $a = (a_1, \cdots, a_n) \in U$, and choose $h > 0$ such that the product open set

$$\prod_{i=1}^{n} N_h(a_i) \quad \subset \quad U \; .$$

If $(x_1, \cdots, x_n) \in U$, then by the preceding sentence we have

$$p(x_1, \cdots, x_n) \quad = \quad \sum_{j=0}^{d} q_j(x_1, \cdots, x_{n-1})\, x_n^j \ .$$

Then for each fixed choice of $(x_1, \cdots, x_{n-1}) \in \prod_{i=1}^{n-1} N_h(a_i)$, the polynomial

$$f(t_n) \quad = \quad p(x_1, \cdots, t_n) \quad = \quad \sum_{j=0}^{d} q_j(x_1, \cdots, x_{n-1})\, t_n^j$$

is zero whenever $t_n \in N_h(a_n)$, and since the proposition is known for polynomials in one indeterminate this polynomial must be zero. Therefore we know that $q_j(x_1, \cdots, x_{n-1}) = 0$ for all j and all $(x_1, \cdots, x_{n-1}) \in \prod_{i=1}^{n} N_h(a_i)$. We can now apply the induction hypothesis to conclude that q_j is the zero polynomial for each j, and this in turn implies that $p = 0$.■

(ii) Let U be a nonempty metrically open open subset of $\mathbb{F}^n$. Then by the first part of the result the open set U is not contained in $V(p)$, which means that $U - V(p)$ is not empty. Since this is one definition or characterization of a dense subset, the conclusion of *(ii)* follows.■