CONGRUENCE AND SIMILARITY CLASSIFICATION

OF REAL HYPERQUADRICS

We shall begin by recalling some material from `quadrics1.pdf`, which gave the classification of hyperquadrics in $\mathbb{R}^n$ and $\mathbb{C}^n$ up to affine equivalence:

CONGRUENCE OR ISOMETRY CLASSIFICATION PROBLEM. *Given two affine hyperquadrics Σ_1 and Σ_2 in $\mathbb{R}^n$, is there a congruence or isometry $\mathbf{T}$ from $\mathbb{R}^n$ to itself mapping Σ_1 onto Σ_2?*

There are two possible versions of the question. An arbitrary isometry $\mathbf{T}$ from $\mathbb{R}^n$ to itself is an affine transformation of the form $\mathbf{T}(\mathbf{x}) = P\,\mathbf{x} + \mathbf{q}$, where P is some orthogonal matrix (*i.e*, ${}^{\mathbf{T}}P = P^{-1}$ and $\mathbf{q}$ is some vector); in some writings, the term "congruence" refers to an arbitrary isometry, while in others it refers to an isometry for which $\det P = +1$. In this document we shall be interested mainly in the less restrictive (first) option.

We also have the following closely related question:

SIMILARITY CLASSIFICATION PROBLEM. *Given two affine hyperquadrics Σ_1 and Σ_2 in $\mathbb{R}^n$, is there a similarity transformation $\mathbf{T}$ from $\mathbb{R}^n$ to itself mapping Σ_1 onto Σ_2?*

A similarity transformation $\mathbf{T}$ from $\mathbb{R}^n$ to itself is an affine transformation of the form $\mathbf{T}(\mathbf{x}) = P\,\mathbf{x} + \mathbf{q}$, where P is is a positive multiple of some orthogonal matrix (the *ratio of similitude*) and $\mathbf{q}$ is some vector. Note that an isometry is a similarity transformation for which the ratio of similitude is $+1$.

We can summarize the relationships among the affine, similarity and congruence classification problems as follows:

(1) *Congruent or isometric affine hyperquadrics are similar.*

(2) *Similar affine hyperquadrics are affinely equivalent.*

In contrast to the topological classification of hyperquadrics in `quadrics2.pdf`, a complete discussion of the classification up to congruence or similarity only requires input from a second course in linear algebra.

The congruence or isometry classification

The discussion of classification up to congruence strongly resembles the corresponding discussion for affine equivalence in several basic ways. In particular, we have the following algebraic result:

TRANSFORMATION FORMULA. *Let Σ be a hypersurface in $\mathbb{R}^n$ defined by a nontrivial second degree equation*

$$f(\mathbf{x}) \quad = \quad {}^{\mathbf{T}}\mathbf{x}\,A\,\mathbf{x} \;+\; 2\cdot{}^{\mathbf{T}}\mathbf{b}\,\mathbf{x} \;+\; c \quad = \quad 0$$

and let G be an isometry of $\mathbb{R}^n$. Then $G[\Sigma]$ is defined by the quadratic equation

$${}^{\mathbf{T}}\mathbf{x}\,{}^{\mathbf{T}}P\,A\,P\,\mathbf{x} \;+\; \left(2\,{}^{\mathbf{T}}P\,(A\,\mathbf{q}+\mathbf{b})\right)\cdot\mathbf{x} \;+\; \left({}^{\mathbf{T}}\mathbf{q}\,A\,\mathbf{q} \;+\; \mathbf{b}\cdot\mathbf{q} \;+\; c\right)$$

where $G^{-1}(\mathbf{x})$ is given by $P\,\mathbf{x}+\mathbf{q}$ for a suitable orthogonal matrix P and vector $\mathbf{q}$.

We know that G^{-1} can be written in the specified form because it is (the inverse of) an isometry.

Proof. If Σ is defined by the quadratic equation $f(\mathbf{x}) = 0$, then we have $\mathbf{x} \in G[\Sigma]$ if and only if $G^{-1}(\mathbf{x}) \in \Sigma$, which holds if and only if $f \circ G^{-1}(\mathbf{x}) = 0$, and hence we can take the defining equation for $G[\Sigma]$ to be $h = f \circ G^{-1}$. If we expand this as in the discussion of the affine classification, we obtain the displayed formula for an equation defining $G[\Sigma]$.■

As in the affine case, we now wish to choose P and $\mathbf{q}$ so that a defining equation for $G[\Sigma]$ takes a simple form. The crucial result for this purpose is the following basic theorem from linear algebra:

FUNDAMENTAL THEOREM ON REAL SYMMETRIC MATRICES. *If A is a real symmetric matrix then there is an orthogonal matrix P such that ${}^{\mathbf{T}}P\,A\,P$ is diagonal.*

One proof of this result is given on pages 51–52 of the following online document:

`http://math.ucr.edu/~res/math132/linalgnotes.pdf`

If we combine this with the methods employed for real affine hyperquadrics (using translations to eliminate some first degree terms), we obtain the following result:

PRELIMINARY CLASSIFICATION UP TO CONGRUENCE. *Let Σ be a hyperquadric in $\mathbb{R}^n$. There is an isometry G such that $G[\Sigma]$ is defined by an equation from the following list:*

$$\sum_{j=1}^{r} d_j\, x_j^2 \;=\; 0 \qquad (1 \;\leq\; r \;\leq\; n)$$

$$\sum_{j=1}^{r} d_j\, x_j^2 \;+\; b\, x_{r+1} \;=\; 0 \qquad (1 \;\leq\; r \;<\; n)$$

$$\sum_{j=1}^{r} d_j\, x_j^2 \;+\; c \;=\; 0 \qquad (1 \;\leq\; r \;\leq\; n)$$

In these defining equations, each real coefficient d_j is nonzero and the terms b and c are also nonzero when they appear.

Following the notational conventions in the affine case, we shall say that these respective possibilities for defining equations are Type $(\mathbf{I}.r)$, Type $(\mathbf{II}.r)$ and Type $(\mathbf{III}.r)$ respectively.

Proof. In many respects the approach is similar to the derivation of the classification up to affine equivalence, so we shall concentrate on what modifications are needed. Assume that the defining equation for Σ is given in the usual form

$$f(\mathbf{x}) \;=\; {}^{\mathbf{T}}\mathbf{x}\,A\,\mathbf{x} \;+\; 2\cdot{}^{\mathbf{T}}\mathbf{b}\,\mathbf{x} \;+\; c \;=\; 0$$

where A is symmetric and nonzero.

First Step. The Fundamental Theorem on Real Symmetric Matrices implies that there is an orthogonal matrix P such that $D \;=\; {}^{\mathbf{T}}P\,A\,P$ is diagonal. If we now take the isometry $G(\mathbf{x}) = P^{-1}\,\mathbf{x}$, then by the Transformation Formula G sends Σ to another hyperquadric Σ^* with a defining equation of the form

$$f^*(\mathbf{x}) \;=\; {}^{\mathbf{T}}\mathbf{x}\,D\,\mathbf{x} \;+\; 2\cdot{}^{\mathbf{T}}\mathbf{b}^*\,\mathbf{x} \;+\; c^* \;=\; 0$$

where D is a diagonal matrix. Let r be the number of nonzero entries down the diagonal of D (so that r is the rank of the original symmetric matrix A).

Second Step. Assuming the defining equation for Σ is given as in the preceding sentence, we shall find an isometry G which permutes the standard unit vectors so that $G[\Sigma]$ is defined by an equation of the form

$$0 \;=\; h(\mathbf{x}) \;=\; {}^{\mathbf{T}}\mathbf{x}\, D_0\, \mathbf{x} \;+\; \mathbf{b}_0 \cdot \mathbf{x} \;+\; c$$

where D_0 is also a diagonal matrix, but its nonzero entries occur precisely in the first r rows and columns.

Let $J \subset \{1, \cdots, n\}$ be the set of all j such that $d_j \neq 0$, let σ be a permutation of $\{1, \cdots, n\}$ which sends J to $\{1, \cdots, r\}$, and let P be the matrix whose j^{th} column is $\mathbf{e}_{\sigma(j)}$. Since the columns of P form an orthonormal basis for $\mathbb{R}^n$, the matrix P is orthogonal. Direct computation shows that $D_0 = {}^{\mathbf{T}}P\, D\, P$ is also a diagonal matrix and its j^{th} diagonal entry is $d_{\sigma(j)}$. Therefore by construction the nonzero entries of D_0 are in the first r rows and columns. If we now choose a linear isometry G such that $G^{-1}(\mathbf{x}) = P\,\mathbf{x}$, then G satisfies the desired conditions.

For the rest of the proof, *assume that Σ is defined by where D is diagonal and its nonzero diagonal entries appear in the first r positions.*

Third Step. Thus far we have shown that Σ is congruent to a hyperquadric in which the second degree terms have the desired form, so we now have to show that the first degree terms and constant term can be simplified if we replace Σ by $G[\Sigma]$ for some isometry G. Assuming that Σ is defined by an equation of the type described in the preceding paragraph, we shall first find a translation G such that $G[\Sigma]$ is defined by an equation of the form

$$0 \;=\; h_1(\mathbf{x}) \;=\; {}^{\mathbf{T}}\mathbf{x}\, D\, \mathbf{x} \;+\; \mathbf{b}_1 \cdot \mathbf{x} \;+\; c_1$$

where the the conditions of the First Step are still satisfied, and in addition the first r coordinates of $\mathbf{b}_1$ are zero.

Define a vector $\mathbf{k}$ by the condition

$$k_j \;=\; \frac{b_j}{2 \cdot \lambda_j} \qquad \text{if} \qquad j \;\leq\; r$$

and $k_j = 0$ if $j > r$ (the latter condition is taken to be vacuously true if $r = n$). If we define $G(\mathbf{x}) = \mathbf{x} + \mathbf{k}$, then $G^{-1}(\mathbf{x}) = \mathbf{x} - \mathbf{k}$, so that $f \circ G^{-1}(\mathbf{x})$ can be written in the form

$$0 \;=\; \sum_{j \leq r} \lambda_j (x_j - k_j)^2 \;+\; \sum_{j=1}^{n} b_j (x_j - k_j) \;+\; c \;=\; \sum_{j \leq r} \lambda_j\, x_j^2 \;+\; \sum_{j=r}^{n} b_j x_j \;+\; c_1$$

where once again a precise description of the constant term c_1 is unimportant. This equation has the desired form provided we define $\mathbf{b}_1$ so that its first r coordinates are zero and its final $n - r$ coordinates are equal to the corresponding coordinates for $\mathbf{b}$.

For the rest of the proof we shall add an assumption that the first r coefficients in the first degree term of are zero as in the conclusion of this step.

Fourth Step. We need one more simplification; namely, we want a form in which the lower degree term is a monomial of degree 1 or 0.

Assuming the defining equation satisfies the conditions of the Third Step, we shall find a linear isometry G such that $G[\Sigma]$ is defined by an equation of the form

$$0 \quad = \quad h(\mathbf{x}) \quad = \quad {}^{\mathbf{T}}\mathbf{x}\, D\, \mathbf{x} \; + \; \mathbf{b}_2 \cdot \mathbf{x} \; + \; c_2$$

where the the conditions of the Third Step are still satisfied, and in addition one of the following holds:

(i) We have $r = n$, so that $\mathbf{b}_2 = 0$.

(ii) We have $r < n$, and $\mathbf{b}_2$ is a (possibly zero) multiple of the unit vector $\mathbf{e}_{r+1}$.

If $r = n$ then there is nothing to prove, so assume that $r < n$. Likewise, if $r < n$ and $\mathbf{b}_1 = \mathbf{0}$ then there is nothing to prove, so assume that $\mathbf{b}_1$ is nonzero. If the latter holds, let $\mathbf{v}$ be the unit vector $|\mathbf{b}_1|^{-1} \cdot \mathbf{b}_1$; by construction the first r coordinates of $\mathbf{v}$ are zero.

Let P be an orthogonal matrix such that $P\,\mathbf{e}_j = \mathbf{e}_j$ for $j \leq r$ and $P\,\mathbf{v} = \mathbf{e}_{r+1}$. Since the nonzero entries of D are concentrated in the $r \times r$ matrix in the upper left hand corner of D, it follows that $D \; = \; {}^{\mathbf{T}}\!P\, D\, P \; = \; P\, D\, {}^{\mathbf{T}}\!P$. If we now define a linear isometry G using the orthogonal matrix P, then G^{-1} is defined by $P^{-1} = {}^{\mathbf{T}}\!P$, and hence $G[\Sigma]$ is defined by the equation(s)

$$0 \quad = \quad h(\mathbf{x}) \quad = \quad {}^{\mathbf{T}}\mathbf{x}\, D\, \mathbf{x} \; + \; 2\,\mathbf{b} \cdot \left({}^{\mathbf{T}}\!P\, \mathbf{x}\right) \; + \; c_1 \quad =$$

$${}^{\mathbf{T}}\mathbf{x}\, D\, \mathbf{x} \; + \; 2\,(P\,\mathbf{b}_1) \cdot \mathbf{x} \; + \; c_1 \quad = \quad {}^{\mathbf{T}}\mathbf{x}\, D\, \mathbf{x} \; + \; 2\,|\mathbf{b}_1|\,\mathbf{e}_{r+1} \cdot \mathbf{x} \; + \; c_1$$

which yield the desired expression for an equation defining $G[\Sigma]$.

Completion of the proof. The preceding four steps imply that if Σ is a hyperquadric in $\mathbb{R}^n$, then Σ is congruent to an object of the same type which is defined by an equation of the form

$$0 \quad = \quad f(\mathbf{x}) \quad = \quad {}^{\mathbf{T}}\mathbf{x}\, D\, \mathbf{x} \; + \; \mathbf{b} \cdot \mathbf{x} \; + \; c$$

where A is a diagonal matrix whose nonzero entries are in an $r \times r$ submatrix at the upper right corner, and if $r < n$ then $\mathbf{b}$ is a (possibly zero) multiple of the unit vector $\mathbf{e}_{r+1}$. In order to complete the proof we need to show that we can find a congruent object which satisfies an equation of this type which either has no constant term or has no first degree term (or both), so assume that both the first degree and constant term in the displayed equation are nonzero. Let $G(\mathbf{x})$ be defined by the translation $\mathbf{x} + L\,\mathbf{e}_{r+1}$, where

$$L \quad = \quad \frac{c}{|\mathbf{b}|}$$

so that $G^{-1}(\mathbf{x}) \;=\; \mathbf{x} - d\,\mathbf{e}_{r+1}$. Then it follows that $G[\Sigma]$ is defined by $0 \;=\; f \circ G^{-1}(\mathbf{x})$, where the right hand side is given by

$${}^{\mathbf{T}}\mathbf{x}\, D\, \mathbf{x} \; + \; \mathbf{b} \cdot \mathbf{x}$$

because $A\,\mathbf{e}_{r+1} = \mathbf{0}$. Therefore we have simplified the lower degree terms to a monomial, and as noted before this completes the proof.■

Statement and proof of the main result

Since congruence implies affine equivalence, it follows that if two hyperquadrics defined by the displayed equations are equivalent, then the two equations must have the same type as described in the preliminary classification. However, two distinct equations of the same type sometimes determine congruent hyperquadrics; for example, if $\{d_1^*, \cdots, d_r^*\}$ is a permutation of $\{d_1, \cdots, d_r\}$, then the hyperquadrics defined by

$$\sum_{j=1}^{r} d_j\, x_j^2 \; + \; c \;\; = \;\; 0 \qquad \text{and} \qquad \sum_{j=1}^{r} d_j^*\, x_j^2 \; + \; c \;\; = \;\; 0$$

are congruent (the linear isometry can be chosen as one which permutes the first r coordinates). Clearly we need to find some conditions on the sequence of coefficients d_j (and possibly b or c) which guarantee that every reasonable hyperquadric is equivalent to exactly one such that the coefficients of the defining equation satisfy the appropriate conditions. We shall follow the approach in Section 9.3 of the book by A. Reventós Tarrida cited in `quadrics1.pdf`. In the latter, the appropriate sequences are said to be *well ordered*, but since that term has an extremely well-established mathematical meaning we shall use the term **conveniently ordered** instead.

Definition. We say that the nonzero real numbers $d_1 \geq \cdots \geq d_r$ are *conveniently ordered* if either of the following two conditions is satisfied:

1. There are more positive d_i's than negative d_i's.
2. The number of positive d_i's is equal to the number of negative d_i's and in the lexicographic ordering for finite sequences of real numbers we have $(d_1, \cdots, d_s) \geq (-d_r, \cdots, -d_{s+1})$, where $s = r/2$.

Recall that if L is a linearly ordered set, then the **lexicographic** (or **dictionary**) ordering of $\prod^s L$ (a product of s copies of L with itself) is defined by $(a_1, \cdots, a_s) < (c_1, \cdots, c_s)$ if and only if $a_i = c_i$ for $i \leq t$ for some t satisfying $0 \leq t < r$ and $a_{t+1} < c_{t+1}$. It is an elementary exercise to verify that if L is a linearly ordered set then the lexicographic ordering on $\prod^s L$ is also a linear ordering.

The reasons for introducing convenient orderings are apparent in the following result:

LEMMA. (compare Reventós Tarrida, Lemma 9.9, p. 290) *Given real numbers $d_1 \geq \cdots \geq d_r$, then either the sequence $d_1 \geq \cdots \geq d_r$ is conveniently ordered or else the sequence $-d_r \geq \cdots \geq -d_1$ is conveniently ordered. Furthermore, if both sequences are conveniently ordered, then they are equal.*

Since the sequence $\{d_1, \cdots, d_r\}$ can be rearranged so that $d_1 \geq \cdots \geq d_r$, it follows that *either the sequence or its negative can be rearranged to obtain a conveniently ordered sequence.*

Proof. If the number of d_j's with $d_j > 0$ is different from the number of d_j's with $d_j < 0$, then exactly one of the two orderings has more positive terms than negative ones, and hence either the original sequence or its negative is conveniently ordered, and the other sequence is not. If there are as many positive d_j's as negative ones, then since the lexicographic ordering is linear we know that either $(d_1, \cdots, d_s) \geq (-d_r, \cdots, -d_{s+1})$ or vice versa; furthermore, if both the original sequence and its negative satisfy this then we have both

$$(d_1, \cdots, d_s) \;\; \geq \;\; (-d_r, \cdots, -d_{s+1}) \quad \text{and} \quad (-d_r, \cdots, -d_{s+1}) \;\; \geq \;\; (d_1, \cdots, d_s)$$

which implies that the two sequences are equal.■

COROLLARY. *Given a sequence of real numbers $\mathbf{d} = \{d_1, \cdots, d_r\}$ there is some $\varepsilon = \pm 1$ such that $\varepsilon\, \mathbf{d}$ has a conveniently ordered rearrangement, and this rearrangement is unique. Furthermore, if both $-\mathbf{d}$ and $\mathbf{d}$ have conveniently ordered rearrangements, then they are equal.*■

Note. A rearrangement $\mathbf{d}^*$ of $\mathbf{d}$ is given by applying a permutation σ of $\{1, \cdots, r\}$ to the sequence:

$$d_i^* \quad = \quad d_{\sigma(i)}$$

However, two distinct permutations may yield the same rearrangement; for example, if all the values of the sequence are equal to 1, then then we always have $\mathbf{d}^* = \mathbf{d}$. We are now ready to state the main theorem on congruence classification of hyperquadrics.

CONGRUENCE CLASSIFICATION THEOREM . *If Σ is a nonempty quadric in $\mathbb{R}^n$ with at least one nonsingular point, then Σ is congruent to one and only one of the quadrics given by the following equations:*

(I.r.d) $x_1^2 + d_2 x_2^2 + \cdots d_r x_r^2 = 0$, *where* $0 \leq r \leq n$ *and* $(1 = d_1, \cdots, d_r)$ *is conveniently ordered.*

(II.r.d) $d_1 x_1^2 + d_2 x_2^2 + \cdots d_r x_r^2 + x_{r+1} = 0$, *where* $0 \leq r < n$ *and* $(d_1, \cdots, d_r)$ *is conveniently ordered.*

(III.r.d) $d_1 x_1^2 + d_2 x_2^2 + \cdots d_r x_r^2 + 1 = 0$, *where* $0 \leq r \leq n$ *and* $d_1 \geq \cdots \geq d_r$.

The notation for the equations follows the earlier conventions for I.r, II.r and III.r, with $\mathbf{d}$ denoting the sequence $(d_1, \cdots, d_r)$.

Proof. We shall first prove that an arbitrary hyperquadric Σ is congruent to an example described in the theorem, and it will suffice to do this for the hyperquadrics listed in the preliminary classification result. The arguments are slightly different for each of the three types.

For equations of Type I, multiply the equation by $\varepsilon = \pm 1$ so that the sequence of coefficients $\mathbf{d}$ can be conveniently ordered, and then take a linear isometry which permutes the first r coordinates so that the rearranged sequence of coefficients $\mathbf{d}^*$ is conveniently ordered. If we apply the isometry to Σ, we obtain a hyperquadric which is defined by an equation similar to Type **(I.r.$\mathbf{d}^*$)**, except that d_1^* is not necessarily equal to $+1$. However, since there is at least one nonsingular point the sequence of coefficients $\mathbf{d}^*$ is nonzero, and this implies that d_1^* must be positive because the sequence is conveniently ordered. Since a positive multiple of a conveniently ordered sequence is conveniently ordered, we can now divide by d_1^* to obtain a defining equation for the new hyperquadric which is of Type **(I.r.$\mathbf{d}^*$)**.

For equations of Type II, we can proceed similarly to obtain a new equation

$$\sum_{j=1}^{r} d_j^* \, x_j^2 \; + \; b\, x_{r+1} \quad = \quad 0 \qquad (b \neq 0)$$

for which the rearranged sequence of coefficients $\mathbf{d}^*$ is conveniently ordered. If we divide by $|b|$, we obtain another equation of the same type such that the coefficient of x_{r+1} is ± 1, and if we now apply the isometry P defined on the standard unit vectors by $P\mathbf{e}_j = \mathbf{e}_j$ if $j \neq r+1$ and $P\mathbf{e}_{r+1} = -\mathbf{e}_{r+1}$, then we obtain a new hyperquadric again of the same type such that the coefficient of x_{r+1} is $+1$, so that the new hyperquadric has a defining equation of Type **(II.r.$\mathbf{d}^*$)**.

For equations of Type III, we are only assuming that the sequence of coefficients $\mathbf{d}$ is in nonincreasing order, and there is no hypothesis that either $d_1 \geq \cdots \geq d_r$ or $-d_r \geq \cdots \geq -d_1$ is conveniently ordered (although we know that at least one of them is). In this case, we first divide by c to obtain a new equation in which the constant term is $+1$, and after this we a take linear isometry which permutes the first r coordinates so that the rearranged sequence of coefficients $\mathbf{d}^*$

satisfies $d_1^* \geq \cdots \geq d_r^*$. If we apply this isometry to the hyperquadric, we obtain a new one with a defining equation of Type **(III.r.d*)**.

Summarizing, *we have shown that every hyperquadric is congruent to an example listed in the theorem.* — To complete the proof, we need to show that a such a hyperquadric is congruent to only one such example.

Suppose now that we are given two hyperquadrics Σ and Σ^* in $\mathbb{R}^n$ which are congruent; since congruent objects are affinely equivalent, it follows that both hyperquadrics are defined by equations of Type N.r, for some fixed r and N (which is I, II or III). Take the defining equations of the hyperquadrics to be

$$(\star) \qquad \sum_{j=1}^{r} d_j\, x_j^2 \;+\; b\, x_{r+1} \;+\; c \;=\; 0\,, \qquad \sum_{j=1}^{s} d_j^*\, x_j^2 \;+\; b^*\, x_{s+1} \;+\; c^* \;=\; 0$$

in coordinates (with the usual convention that at least one of b and c must be zero) and equivalently by

$${}^{\mathbf{T}}\mathbf{x}\, D\, \mathbf{x} \;+\; (b\, \mathbf{e}_{r+1}) \cdot \mathbf{x} \;+\; c \;=\; 0\,, \qquad {}^{\mathbf{T}}\mathbf{x}\, D^*\, \mathbf{x} \;+\; (b^*\, \mathbf{e}_{r+1}) \cdot \mathbf{x} \;+\; c^* \;=\; 0$$

in vector/matrix form, where $\mathbf{e}_{r+1}$ is the usual unit vector whose j^{th} coordinate is 1 if $j = r+1$ and 0 otherwise. If we write the inverse of the isometry G in the usual vector form $G^{-1}(\mathbf{x}) = P\,\mathbf{x} + \mathbf{q}$, then by the Transformation Formula the image $G[\Sigma]$ is defined by the equation

$${}^{\mathbf{T}}\mathbf{x}\, {}^{\mathbf{T}}P\, D\, P\, \mathbf{x} \;+\; \left({}^{\mathbf{T}}P\, (2\, D\, \mathbf{q} + b\, \mathbf{e}_{r+1})\right) \cdot \mathbf{x} \;+\; \left({}^{\mathbf{T}}\mathbf{q}\, D\, \mathbf{q} \;+\; (b\, \mathbf{e}_{r+1}) \cdot \mathbf{q} \;+\; c\right)$$

and by the uniqueness results for defining equations in `quadrics1.pdf` this equation must be equal to k times the previous given equation for Σ^*, where k is some nonzero real number. Note that *this is the point in the proof where we need to assume that each hyperquadric has at least one nonsingular point.*

One important consequence of the preceding discussion is that the diagonal matrices D and D^* are related by an equation of the form

$${}^{\mathbf{T}}P\, D\, P \;=\; k \cdot D^*$$

for some orthogonal matrix P and some nonzero scalar k. Since the eigenvalues of a diagonal matrix are its diagonal entries, the equation ${}^{\mathbf{T}}P\, D\, P = k \cdot D^*$ implies that the diagonal entries of D must be a rearrangement of k times the diagonal entries of D^*. The matrix equation also has the following important implication:

> CLAIM. *The matrix P sends the vector subspaces V and W spanned by $\{\mathbf{e}_1, \cdots, \mathbf{e}_r\}$ and $\{\mathbf{e}_{r+1}, \cdots, \mathbf{e}_n\}$ into themselves.*
>
> PROOF OF CLAIM. Since the vector spaces in question are orthogonal complements of each other and P is an orthogonal matrix, it suffices to prove that P maps W into itself. Since D and $D*$ are diagonal matrices whose nonzero entries are in the first r positions, it follows that W is the kernel of both D and D^*. The matrix identity connecting these matrices can be rewritten as $DP = kPD^*$, so $\mathbf{w} \in W$ implies $D^*\,\mathbf{w} = \mathbf{0}$, which translates to $DP\,\mathbf{w} = kPD^*\,\mathbf{w} = kP\,\mathbf{0} = \mathbf{0}$, so that $P\,\mathbf{w}$ lies in the kernel of D, which is W.

The argument now splits into three cases, depending upon whether the defining equations for the two hyperquadrics in $(\star)$ are of Type I, II or III.

The Type **I** case. Suppose that we have two hyperquadrics on the list which are defined by equations of Type I and are congruent; we assume the equations are given as in ($\star$) above. Let $\mathbf{d}$ and $\mathbf{d}^*$ denote the sequences of diagonal entries for D and D^* respectively. Since we are considering equations of Type I, both $\mathbf{d}$ and $\mathbf{d}^*$ are conveniently ordered sequences. Furthermore, the equation ${}^{\mathbf{T}}\!P\, D\, P = k \cdot D^*$ implies that $\mathbf{d}$ and $k\,\mathbf{d}^*$ are rearrangements of each other.

The crucial step in the argument for the Type I case will be to prove that these two sequences are equal. There are two cases, depending upon whether k is positive or negative. Since a positive multiple of a conveniently ordered sequence is conveniently ordered, it follows that $k\,\mathbf{d}^*$ is conveniently ordered if $k > 0$, and since every sequence has at most one conveniently ordered rearrangement it follows that the sequences $\mathbf{d}$ and $k\,\mathbf{d}^* = |k|\,\mathbf{d}^*$ must be equal, proving the statement at the end of the previous paragraph when $k > 0$.

On the other hand, if $k < 0$, then $\mathbf{d}$ is the convenient reordering of $k\,\mathbf{d}^*$, and since $\mathbf{d}^*$ itself is conveniently ordered, it follows that $d_j = k\,d^*_{r+1-j}$ for all j. By the discussion in the previous paragraph we also know that $|k|\,\mathbf{d}^*$ is conveniently ordered, and since this sequence and its negative both have convenient rearrangements we know that these rearrangements must satisfy

$$|k|\,d^*_j \quad = \quad k\,d^*_{r+1-j} \quad = \quad -|k|\,d^*_{r+1-j}$$

for all j. If we cancel $|k|$ from the first and third equation, we find that $d^*_j = -d^*_{r+1-j}$, and if we combine this with the previous equation we see that $d_j = |k|\,d^*_j$ for all j, which is the same conclusion we established in the preceding paragraph for the case $k > 0$. Note that *none of the preceding discussion involves the normalization conditions* $d_1 = d^*_1 = 1$.

Applying the normalization condition in the preceding sentence, we see that $1 = d_1 = |k|\,d^*_1 = |k|$, which in turn implies that $d_j = d^*_j$ for defining equations of Type I.

The Type **II** case. The arguments in the Type I case imply that $d_j = |k|\,d^*_j$ for all j regardless of whether k is positive or negative, and as in the Type I case it will suffice to show that $|k| = 1$. Not surprisingly, this will require some analysis of first degree terms.

Suppose that the Galilean transformation G with $G^{-1}(\mathbf{x} = P\mathbf{x} + \mathbf{q}$ sends the first hyperquadric Σ into the second hyperquadric Σ^*. Then the Transformation Formula and the results on two equations defining the same hyperquadric imply that

$$2\,{}^{\mathbf{T}}\!P\,D\,\mathbf{q} \;+\; {}^{\mathbf{T}}\!P\,\mathbf{e}_{r+1} \quad = \quad k\,\mathbf{e}_{r+1}$$

for the same $k \neq 0$ as before. In the notation of the claim verified above, we know that $D\,\mathbf{q} \in V = W^{\perp}$, and since the transpose of an orthogonal matrix is its inverse we know that $D\mathbf{e}_{r+1} \in W$. Since the right hand side of the displayed equation lies in W, this means that the first term on the left hand side of the display must vanish. Now the matrix ${}^{\mathbf{T}}\!P$ is orthogonal (the traspose=inverse of an orthogonal matrix is orthogonal) so the length of the vector on the left hand side of the displayed equation is equal to 1; since the length of the vector on the right is $|k|$, it follows that $|k| = 1$, and as in the argument for Type I this implies that $d_j = d^*_j$ for all j if we have two congruent hyperquadrics on the list which are defined by equations of Type II.

The Type **III** case. Note that there is no assumption that $\mathbf{d}$ or $\mathbf{d}^*$ be conveniently ordered, but we are assuming that both sequences are nondecreasing. For these examples we must examine both the constant terms and the first degree terms. Specifically, the Transformation Formula and the results on two equations defining the same hyperquadric imply that

$$2\,{}^{\mathbf{T}}\!P\,D\,\mathbf{q}\cdot\mathbf{x} \quad = \quad 0 \quad \text{and} \quad {}^{\mathbf{T}}\mathbf{q}\,{}^{\mathbf{T}}\!P\,A\,P\,\mathbf{q} \;+\; 1 \quad = \quad k$$

for the same $k \neq 0$ as before. The invertibility of P (and also of its transpose) implies that $D\,\mathbf{q} = \mathbf{0}$, so that the first degree term vanishes and similarly we have

$$^{\mathbf{T}}\mathbf{q}\,^{\mathbf{T}}P\,D\,P\,\mathbf{q} \quad = \quad 0$$

which implies that $k = 1$. Since $\mathbf{d}$ is a nonincreasing rearrangement of $k\,\mathbf{d}^*$, the equation $k = 1$ implies that $\mathbf{d} = \mathbf{d}^*$, which completes the proof if we have two congruent hyperquadrics on the list which are defined by equations of Type III.∎

The similarity classification

A classification up to similarity follows from the classification up to congruence with only a little additional work.

SIMILARITY CLASSIFICATION THEOREM . *If Σ is a nonempty quadric in $\mathbb{R}^n$ with at least one nonsingular point, then Σ is congruent to one and only one of the quadrics given by the following equations:*

(I.r.d) $x_1^2 + d_2 x_2^2 + \cdots\ d_r x_r^2 = 0$, *where* $0 \leq r \leq n$ *and* $(1 = d_1, \cdots, d_r)$ *is conveniently ordered.*

(II.r.d) $x_1^2 + d_2 x_2^2 + \cdots\ d_r x_r^2 + x_{r+1} = 0$, *where* $0 \leq r < n$ *and* $(1 = d_1, \cdots, d_r)$ *is conveniently ordered.*

(III.r.d) $d_1 x_1^2 + d_2 x_2^2 + \cdots\ d_r x_r^2 + 1 = 0$, *where* $0 \leq r \leq n$ *and* $\pm 1 = d_1 \geq \cdots \geq d_r$.

The notation for the equations follows the conventions in the Congruence Classification Theorem.

Proof. We begin by reviewing some simple but important properties involving similar geometrical figures (*i.e.*, subsets of $\mathbb{R}^n$):

> *Two figures X and Y are congruent if and only if they are similar such that the ratio of similitude is* 1.
>
> *If there is a similarity from X to Y with ratio of similitude α, then there is a similarity from Y to X with ratio of similitude α^{-1}.*
>
> *If there is a similarity from X to Y with ratio of similitude α and there is a similarity from Y to Z with ratio of similitude β, then there is a similarity from X to Z with ratio of similitude $\beta \cot \alpha$.*

One consequence of these properties is that *if there are similarities from X to Y and from X to Z with the same ratio of similitude α, then Y and Z are congruent.*

One major step in the proof is verifying the following assertion:

> *If Σ is a hyperquadric defined by one of the equations in the list for the Congruence Classification Theorem and $\alpha > 0$, then there is a unique second hyerquadric Σ' defined by one of the equations in the list such that Σ is similar to Σ' with ratio of similitude α.*

The verification of this assertion splits into cases depending upon whether Σ and Σ' are defined by equations of Type I, II or III, and it begins with the observations stated below. In each case, c denotes an arbitrary positive number.

> There is a similarity from the hyperquadric defined by equation **(I.r.d)** to itself with ratio of similitude equal to c.

There is a similarity from the hyperquadric defined by equation **(II.**r**.d)** to the hyperquadric defined by equation **(II.**r**.**c^{-1} **d)** with ratio of similitude equal to c.

There is a similarity from the hyperquadric defined by equation **(III.**r**.d)** to the hyperquadric defined by equation **(III.**r**.**c^{-2} **d)** with ratio of similitude equal to c.

In each case the similarity is the map $\mathbf{y} = \mathbf{T}(\mathbf{x}) = c\,\mathbf{x}$. The derivations in the three cases are similar but not identical.

For hyperquadrics determined by equations of Type I, we have a homogeneity identity

$$f(\,\mathbf{T}(\mathbf{x})\,) \quad = \quad f(c\,\mathbf{x}) \quad = \quad c^2\, f(\mathbf{x})$$

which shows that $f(\,\mathbf{T}(\mathbf{x})\,) = 0$ if and only if $= 0$, so that $\mathbf{T}(\mathbf{x}) = c\,\mathbf{x} \in \Sigma$ if and only if $\mathbf{x} \in \Sigma$. Therefore $T[\Sigma] = \Sigma$.

For hyperquadrics determined by equations of Type II, we have

$$f(\,\mathbf{T}(\mathbf{x})\,) \quad = \quad f(c\,\mathbf{x}) \quad = \quad \sum_{j=1}^{r} d_j\, \left(\frac{y_j^2}{c^2}\right) \ + \ \frac{y_{r+1}}{c}$$

and therefore Σ satisfies **(II.**r**.d)** if and only if $\mathbf{T}[\Sigma]$ satisfies **(II.**r**.**c^{-1}**d)**.

Finally, for hyperquadrics determined by equations of Type III, we have

$$f(\,\mathbf{T}(\mathbf{x})\,) \quad = \quad f(c\,\mathbf{x}) \quad = \quad \sum_{j=1}^{r} d_j\, \left(\frac{y_j^2}{c^2}\right) \ + \ 1$$

and therefore Σ satisfies **(II.**r**.d)** if and only if $\mathbf{T}[\Sigma]$ satisfies **(II.**r**.**c^{-2} **d)**.

The preceding shows that if Σ is defined by the polynomial **(Y.**r**.d)** and $c > 0$. then Σ is similar to the hyperquadric defined by some polynomial **(Y.**r**.d*****)** with ratio of similitude c. In fact, $\mathbf{d}^*$ is unique, for if the hyperquadric defined by both **(Y.**r**.d*****)** and **(Y.**r**.d**$^\#$**)** with ratio of similitude c, then the latter two quadrics are congruent and hence $\mathbf{d}^* = \mathbf{d}^\#$).

The preceding discussion shows that the similarity classes of the representatives for Types II and III in the Congruence Classification Theorem consist of all examples defined by the equations **(Y.**r**.**α**d)**, where α varies over all positive real numbers. To obtain the canonical representatives in the Similarity Classification Theorem, divide $\mathbf{d}$ by $|d_1|$ if $\mathbf{Y} = \mathbf{II}$ or **III**. This completes the proof for Types II and III. Similarly, the preceding discussion shows that the similarity classes of the representatives for Type I in the Congruence Classification Theorem are the same as the congruence classes.∎

Application to conics in $\mathbb{R}^2$

We shall conclude this document with a geometrically explicit description of the congruence and similarity classifications for the standard conics in $\mathbb{R}^2$; namely, the circle, ellipse, parabola and hyperbola. In other words, we want to describe the geometrical properties of these objects which correspond to the coefficients d_j.

The circle can be treated without using much of the machinery developed in this document. On the other hand, we shall assume a fair amount of background involving various points and lines associated to a conic. At the end of this document there are some illustrations of hyperbolas and parabolas which may be useful; also, the online document

`http://www20.csueastbay.edu/library/scaa/files/pdf/Conics.pdf`

gives a one page summary of basic concepts involving conics (however, there are no illustrations).

CIRCLES. In the list of examples from the Congruence Classification Theorem, the circles are defined by equations of the form

$$d_1 x_1^2 \ + \ d_2 x_2^2 \ + \ 1 \quad = \quad 0$$

where $0 > d_1 = d_2$. The diameter of the circle is equal to $2/\sqrt{|d_2|} = 2/\sqrt{|d_1|}$, and the radius is equal to $1/\sqrt{|d_2|} = 1/\sqrt{|d_1|}$.

CLASSIFICATION OF CIRCLES. *Two circles are congruent if and only if they have the same radii, and every circle is similar to every other circle.*

Proof. We shall first prove the statements regarding congruence. The maximum distance between two points of a circle is twice the radius, and if two circles are congruent then the maximum distance between two points in one circle is equal to the maximum distance between two points in one circle. Therefore if two circles are congruent, then twice their radii must be equal, and this means that their radii must be equal. Conversely, if two circles have the same radius and their centers are $\mathbf{p}$ and $\mathbf{q}$, then the translation $\mathbf{T}(\mathbf{x}) = \mathbf{x} + \mathbf{q} - \mathbf{p}$ sends the circle with center $\mathbf{p}$ and radius r to the circle with center $\mathbf{q}$ and radius r.

Suppose now that we have one circle with center $\mathbf{p}$ and radius r, and a second circle with center $\mathbf{q}$ and radius s. Then

$$\mathbf{T}(\mathbf{x}) \quad = \quad s\,r^{-1}\,\mathbf{x} \ + \ \left(\mathbf{q} \ - \ s\,r^{-1}\,\mathbf{p}\right)$$

is a similarity transformation which sends the first circle to the second.■

The discussion of ellipses requires the concepts of major or minor axes. These are the line segments which pass through the center of the ellipse and have minimum and maximum length. As simple examples suggest, these segments meet at the center and are perpendicular to each other.

ELLIPSES. In the list of examples from the Congruence Classification Theorem, the ellipses are defined by equations of the form

$$d_1 x_1^2 \ + \ d_2 x_2^2 \ + \ 1 \quad = \quad 0$$

where $0 > d_1 > d_2$. The lengths of the major and minor axes are equal to $2/\sqrt{|d_1|}$ and $2/\sqrt{|d_2|}$ respectively, and the eccentricity of the ellipse is given by

$$e \quad = \quad \sqrt{1 \ - \ \frac{d_1}{|d_2|}}\ .$$

These lengths can be defined synthetically (*i.e.*, without using coordinates or defining equations) as follows:

LEMMA. *If the ellipse Γ is defined by*

$$\frac{x^2}{a^2} \ + \ \frac{y^2}{b^2} \quad = \quad 1$$

(where $a > b > 0$) then the distance between two points in Γ is less than or equal to $2a$, and equality holds if and only if the two points are $(\pm a, 0)$.

Proof of the Lemma. By the Triangle Inequality, it suffices to show that $\mathbf{v} = (x, y) \in \Gamma$ satisfies $|\mathbf{v}| \leq a$ and equality holds if and only if $y = 0$; it suffices to show that the square of the given inequality holds, and that one has equality in the latter if and only if $y = 0$ (which holds if and only if $x = \pm a$). But the defining equation for Γ shows that if $\mathbf{v} \in \Gamma$ then

$$a^2 \;=\; x^2 \,+\, \frac{a^2y^2}{b^2} \;=\; |\mathbf{v}|^2 \,+\, \left(\frac{a^2}{b^2} \,-\, 1\right) y^2$$

and the coefficient of y^2 in the expression on the right is positive because $a^2 > b^2$, and therefore we have $x^2 + y^2 \leq a^2$, with equality if and only if $y = 0$.■

Once we have the end points of the major axis, we can retrieve the center of the ellipse because it is the midpoint of the line segment joining the end points of the major axis, and we can retrieve the minor axis from the line perpendicular to the major axis at its midpoint by taking the pair of points where this perpendicular meets the ellipse.

CLASSIFICATION OF ELLIPSES. *Two ellipses are congruent if and only if the lengths of their major and minor axes are equal, and two ellipses are similar if and only if the ratios of the lengths of their major to the lengths of their minor axes are equal. Equivalently, two ellipses are similar if and only if their eccentricities are equal.*

Proof. By the congruence classification, the ellipses are congruent to examples defined by the equations

$$d_1x_1^2 \,+\, d_2x_2^2 \,+\, 1 \;=\; 0\,, \qquad d_1^*x_1^2 \,+\, d_2^*x_2^2 \,+\, 1 \;=\; 0$$

and the coefficients of x_2^2 and x_1^2 give the lengths of the major and minor axes respectively; more precisely, the coefficients can be expressed in terms of the lengths and vice versa. The ellipses are congruent if and only if $d_1 = d_1^*$ and $d_2 = d_2^*$, so the condition for congruence reduces to the equality of the lengths of the major and minor axes.

Regarding the classification up to similarity, the classification implies that two ellipses are similar if and only if the ratios d_2/d_1 and d_2^*/d_1^* are equal. As noted before the proof, the square roots of these ratios are the ratios of the lengths of the major to the minor axes, so the ellipses are similar if and only if the length ratios are equal. Furthermore, the eccentricity formula shows that this quantity and the length ratio completely determine each other, so it follows that the two ellipses are similar if and only if they have the same eccentricities.■

PARABOLAS. In the list of examples from the Congruence Classification Theorem, the parabolas are defined by equations of the form

$$d_1x_1^2 \,+\, x_2 \;=\; 0$$

where $d_1 > 0$. The focus (or focal point) or the parabola $y = ax^2/4$ has coordinates $(a, 0)$ (see the illustrations at the end of this document), so for the standard examples as above the focal points have coordinates $(0, -\frac{1}{4}\,d_1)$; note that these parabolas, whose equations can be rewritten in the form $x_2 = -d_1x_1^2$ where $d_1 > 0$, are contained in the lower half plane defined by the inequality $x_2 \leq 0$.

The vertex of a parabola Γ can be defined synthetically as follows: There is a unique line L (the *axis of symmetry*) such that Γ is equal to its own mirror image with respect to L (formally, the mirror image is obtained using the standard reflection through L). There is a unique line M perpendicular to L which meets Γ in exactly one point; this line M is tangent to Γ, and the vertex is the point of contact.

In contrast, a synthetic definition of the focus (or focal point) requires more effort and is better described using concepts from physics. The complement of Γ splits into two connected regions, and exactly one of these regions is convex; for an example whose equation is given as in the Classification Theorem, the convex region is defined by the inequality $d_1 x_1^2 + x_2 < 0$. If a light ray is parallel to the axis of symmetry and aimed at the parabola, then its reflection off the parabola passes through the focus, and hence the focus is the unique point which lies on each of these reflected rays (see the drawing at the end of this document); a mathematical derivation of this property is given in the following online document:

`http://www.analyzemath.com/parabola/parabola_work.html`

For the sake of completeness, the mathematical model for reflection of the light ray is that one has an incoming ray R_1 and an outgoing ray R_2 starting at the point on the parabola, and if T_1 and T_2 denote the tangent rays at the point such that R_1 and T_1 make an acute angle, then the measure of this angle is also equal to the measure of the angle made by R_2 and T_2 (in physical terms, the angle of incidence equals the angle of reflection).

CLASSIFICATION OF PARABOLAS. *Two parabolas are congruent if and only if the distances from their vertices to their focal points are equal, and all parabolas are similar to each other.*

Proof. The result on similarity in the preceding sentence follows because every parabola is similar to the example defined by $x_1^2 + x_2 = 0$ by the Similarity Classification Theorem. Regarding the classification up to congruence, given two parabolas defined by the equations

$$d_1 x_1^2 \ + \ x_2^2 \ = \ 0 \ , \qquad d_1^* x_1^2 \ + \ x_2^2 \ = \ 0$$

the coefficient of x_1 determines the distance between the vertex and the focus and vice versa, so the two parabolas are congruent if and only if these distances are equal.■

HYPERBOLAS. In the list of examples from the Congruence Classification Theorem, the hyperbolas are defined by equations of the form

$$d_1 x_1^2 \ + \ d_2 x_2^2 \ + \ 1 \ = \ 0$$

where $d_1 > 0 > d_2$. The two branches of this hyperbola are defined by $\sqrt{|d_2|}x_2 = \pm\sqrt{1 + d_1 x_1^2}$, the vertices have coordinates $(0, \pm 1/\sqrt{|d_2|})$, and it follows that the distance between the vertices is $2/\sqrt{|d_2|}$, so that d_2 and this distance completely determine each other. The asymptotes of the hyperbola are the lines with equations

$$x_2 \ = \ \pm \sqrt{\frac{d_1}{|d_2|}}\, x_1$$

and the asymptotic cone associated to the hyperbola is the set of points satisfying

$$d_1 x_1^2 \ + \ d_2 x_2^2 \ \leq \ 0 \ .$$

The boundary of this closed region is a pair of vertical angles whose radian measures are given by

$$2 \arctan \sqrt{\frac{|d_2|}{d_1}} \ .$$

These quantities can be defined synthetically as follows: The vertices are the unique pair of points $\mathbf{v}_1$ and $\mathbf{v}_2$ are the unique pair of points on different branches of the hyperbola for which the distance $d(\mathbf{v}_1, \mathbf{v}_2)$ is minimized; in fact, if at least one of these points is not a vertex, then the distance between the first coordinates of the points will be strictly greater than $2/\sqrt{|d_2|}$. Once we have the vertices, we can retrieve the center of the hyperbola because it is the midpoint of the line segment joining the vertices, and the asymptotic cone will be the closure of the set of all points on lines joining points of a hyperbola to its center (as before, see the drawings at the end of this document). Finally, the eccentricity of the hyperbola is given by

$$e \quad = \quad \sqrt{1 \; + \; \frac{|d_2|}{d_1}} \; .$$

CLASSIFICATION OF HYPERBOLAS. *Two hyperbolas are congruent if and only if the distances between their vertices are equal and the radian measures of their asymptotic cones are equal, and two hyperbolas are similar if and only if the ratios of the the radian measures of their asymptotic cones are equal. Equivalently, two hyperbolas are similar if and only if their eccentricities are equal.*

By construction the eccentricity of an ellipse is always strictly between 0 and 1, and the eccentricity of a hyperbola is always strictly greater than 1. Therefore, if we set the eccentricities of a circle and parabola equal to 0 and 1 respectively, we can summarize the similarity classification for conics into a unified statement: *Two standard conics in $\mathbb{R}^2$ are similar if and only if their eccentricities are equal.*

Proof of the classifications of hyperbolas. By the congruence classification, the hyperbolas are congruent to examples defined by the equations

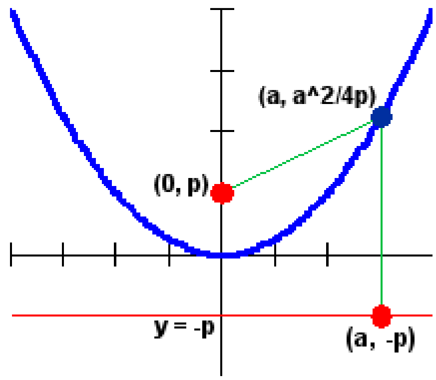
$$d_1 x_1^2 \; + \; d_2 x_2^2 \; + \; 1 \quad = \quad 0 \; , \qquad d_1^* x_1^2 \; + \; d_2^* x_2^2 \; + \; 1 \quad = \quad 0$$

and the coefficients of x_2^2 determine the distances between the vertices (and vice versa). Therefore, if the hyperbolas are congruent then the distances between the vertices are equal. Similarly, the radian measures of the asymptotic cones are completely determined by $d_1/|d_2|$ and vice versa, so if these radian measures are equal if the two hyperbolas are congruent. Conversely, since the radian measure and distance between the vertices determine the coefficients of x_1 and x_2, it follows that if the radian measure and distance between the vertices are equal then the hyperbolas are congruent (to the same standard example).

Regarding the classification up to similarity, the classification implies that two hyperbolas are similar if and only if the ratios d_2/d_1 and d_2^*/d_1^* are equal. As noted before the proof, these ratios are determined by the radian measure of the asymptotic cone and vice versa, so the hyperbolas are similar if and only if the radian measures of the asymptotic cones are equal. Furthermore, the eccentricity formula shows that this quantity and the the radian measure of the asymptotic cone completely determine each other, so it follows that the two hyperbolas are similar if and only if they have the same eccentricities.■

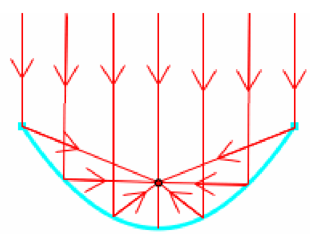
Addendum — Some pictures of conics in the plane

The first picture indicates how one finds the focus of the parabola defined by $y = x^2/4p$.



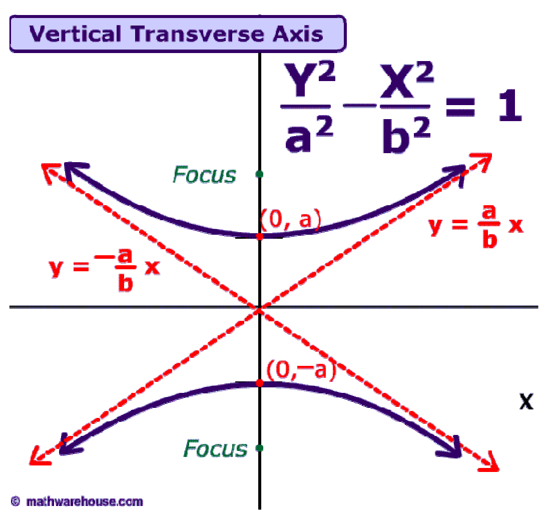
(***Source:*** **http://cs.bluecc.edu/conics/04.shtml**)

The second picture illustrates the reflection property of a parabola**:** Parallel beams of light which are aimed at a parabola will converge at the focus after being reflected**. X — x**



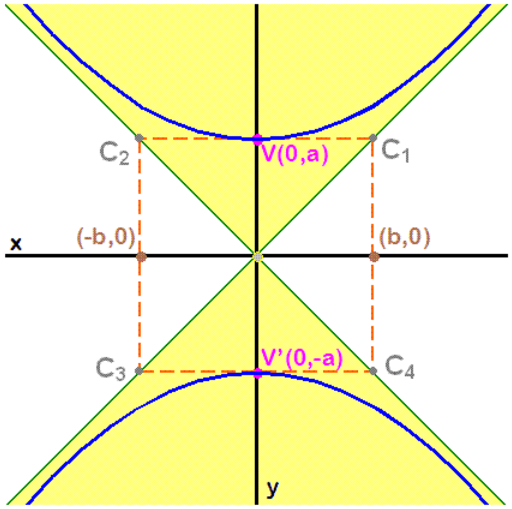
(***Source:*** **http://jwilson.coe.uga.edu/EMAT6680Fa08/Wisdom/EMAT6690/Parabolanjw/reflectiveproperty.htm**)

The third picture illustrates the hyperbola which arises in the congruence classification of conics**,** but with a difference in notation**:** d_1 corresponds to $1/b^2$ and d_2 corresponds to $-1/a^2$.



(***Source:*** **http://www.mathwarehouse.com/hyperbola/graph-equation-of-a-hyperbola.php**)

The final picture illustrates the asymptotic cone of the hyperbola in the third picture (shaded in yellow). The four points C_i on the asymptotes are vertices of a rectangle centered at the origin, and C_1 had coordinates (b, a).



(Source: http://www2.lv.psu.edu/ojj-rcm27/topics/hyperbolas.html)