SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145B — Part 5

Spring 2015

V. Further topics

V.1 : Homotopy and line integrals

Additional exercises

1. (*a*) The open subset described in this part of the exercise is convex, and hence it has a trivial fundamental group. Therefore γ is base point preservingly homotopic to the constant curve K with the same base point. The integrand satisfies the basic partial derivative condition $Q_x = P_y$, and therefore by the results on homotopy invariance the line integrals of γ and K are equal. Since the integral over K is zero by the basic properties of line integrals, the same must be true for the integral over γ.■

(*b*) Let $\delta(t) = \lambda(t) - \gamma(t)$, so that the hypotheses imply that $|\delta| \leq 1$. It follows that if $s \in [0,1]$ then $\lambda_s(t) = \gamma(t) + s\,\delta(t)$ satisfies

$$|\lambda_s(t)| \;=\; |\gamma(t) + s\,\delta(t)| \;\geq\; |\gamma(t)| \;-\; s\,|\delta(t)| \;>\; 1 \;-\; s\,|\delta(t)|$$

and the right hand side is positive because $s \in [0,1]$ and we are assuming that $|\delta| \leq 1$. Therefore the image of the straight line homotopy lies in the complement ofo $\{\mathbf{0}\}$, so that λ and γ are (freely) homotopic in $\mathbb{R}^2 - \{\mathbf{0}\}$ and hence the values of the line integrals over these curves must be equal.■

2. (*a*) We begin with the proof that $\int_\gamma \mathbf{F}\cdot d\mathbf{x} = 0$ for all closed rectifiable curves γ in $\mathbb{R}^2 - \{\mathbf{0}\}$. Let $\mathbf{e}_1 \in \mathbb{R}^2$ be the standard unit vector $(1,0)$; then the basic algebraic properties of line integrals together with homotopy invariance imply that the line integral yields a well defined homomorphism from $\pi_1(\mathbb{R}^2 - \{\mathbf{0}\}, \mathbf{e}_1)$ to the real numbers. This group is infinite cyclic and generated by the class of the closed curve θ. Therefore $\int_\theta \mathbf{F}\cdot d\mathbf{x} = 0$ implies that $\int_\gamma \mathbf{F}\cdot d\mathbf{x} = 0$, at first for all closed rectifiable curves γ which are base point preserving iterated concatenations of θ and $-\theta$, and ultimately for all closed rectifiable curves γ which are base point preserving. Since the forgetful map from $\pi_1(\mathbb{R}^2 - \{\mathbf{0}\}, \mathbf{e}_1)$ to $[S^1, \mathbb{R}^2 - \{\mathbf{0}\}]$ (which sends base point preserving homotopy classes to free homotopy classes) is onto and the line integral depends only on the free homotopy class of a closed curve, it follows that $\int_\gamma \mathbf{F}\cdot d\mathbf{x} = 0$ for all closed rectifiable curves γ in $\mathbb{R}^2 - \{\mathbf{0}\}$.

We now use the preceding paragraph's conclusion to prove the path independence statement. Let $\gamma = \alpha + (-\beta)$, so that γ is a closed rectifiable curve. By the preceding discussion we have

$$0 \;=\; \int_\gamma \mathbf{F}\cdot d\mathbf{x} \;=\; \int_\alpha \mathbf{F}\cdot d\mathbf{x} \;-\; \int_\beta \mathbf{F}\cdot d\mathbf{x}$$

which is equivalent to the path independence equation $\int_\alpha \mathbf{F}\cdot d\mathbf{x} = \int_\beta \mathbf{F}\cdot d\mathbf{x}$.■

(b) Define the function g as indicated in the hint, and define curves $\delta_1(t) = (x+t, y)$, $\delta_2(t) = (x, y+t)$; since (x, y) lies in the open set $\mathbb{R}^2 - \{\mathbf{0}\}$, one can find some $h > 0$ such that both curves $\delta_j(t)$ take values in $\mathbb{R}^2 - -\{\mathbf{0}\}$ for $|t| \leq h$. Therefore, if γ is the curve in the hint, we have

$$g(x+t, y) \quad = \quad \int_{\gamma+\delta_1} \mathbf{F} \cdot d\mathbf{x} \quad = \quad \int_{\gamma} \mathbf{F} \cdot d\mathbf{x} \;+\; \int_{\delta_1} \mathbf{F} \cdot d\mathbf{x}$$

and similarly for $g(x, y+t)$. These immediately yield the following identities:

$$\frac{g(x+t, y) - g(x, y)}{t} \quad = \quad \frac{1}{t} \cdot \int_{\delta_1} \mathbf{F} \cdot d\mathbf{x} \quad = \quad \frac{1}{t} \cdot \int_0^t \; P(x+s, y)\, ds$$

$$\frac{g(x, y) - g(x, y+t)}{t} \quad = \quad \frac{1}{t} \cdot \int_{\delta_2} \mathbf{F} \cdot d\mathbf{x} \quad = \quad \frac{1}{t} \cdot \int_0^t \; Q(x, y+s)\, ds$$

These are true because the dy term vanishes for the horizontal line segment δ_1 and the dx term vanishes for the vertical line segment δ_2. Suppose we now take limits as $t \to 0$, so that the limits of the left and right hand sides are also equal. Then the first expressions show that the first partial derivative of g at (x, y) is equal to $Q(x, y)$, and the second expressions show that the second partial derivative of g at (x, y) is equal to $P(x, y)$.■

3. Suppose first that the condition on partial derivatives is satisfied. Then by homotopy invariance we know that the line integrals along closed rectifiable curves always vanish. Suppose now that the condition on partial derivatives is not satisfied at some point $p_0 \in U$. By continuity there is some $r > 0$ such that $|z - p_0| \leq r$ implies that

$$\Delta(z) \quad = \quad \frac{\partial Q}{\partial x}(z) \;-\; \frac{\partial P}{\partial y}(z)$$

is either greater than or equal to some small positive constant η or is less than or equal to $-\eta$. In either case we have

$$\left| \int_{B(r)} \; \Delta(x, y)\, dx\, dy \right| \quad \geq \quad \eta\, \pi\, r^2 \quad > \quad 0$$

if $B(r)$ is the disk defined by $|z - p_0| \leq r$. Let $\theta(r)$ be the boundary of $B(r)$ parametrized counterclockwise. Then by Green's Theorem the double integral is equal to the line integral $\int_{\theta(r)} \; P\, dx + Q\, dy$, and thus it follows that this line integral is also nonzero.■

V.2 : Graph complexes

Additional exercises

1. The vertices are ordered $n-$tuples whose coordinates are either 0 or 1. For each j there are two possibilities for the coordinate, and these are independent of the previous choices. Therefore it follows that there are 2^n possible $n-$tuples and hence the hypercube has 2^n vertices. If this argument seems too abstract at first, look at the special case $n = 3$ and then at the case $n = 4$.

The edges are pairs of n-tuples which agree in all but one entry. If we fix j such that $1 \leq j \leq n$, then there are 2^{n-1} choices of pairs which have all coordinates equal except for the j^{th} ones. If we let j range from 1 to n, then no two sets of pairs have a pair in common, and the union of all the pairs corresponds to the set of edges. Therefore there must by $n\,2^{n-1}$ distinct edges.■

2. (a) Every edge has two endpoints which are vertices, and no pair of vertices form the endpoints of more than one edge (there might be no edge with the pair as its endpoints, but there cannot be two or more). Therefore the set of edges is in 1–1 correspondence with a subset of the set of pairs of vertices. If V is the total number of vertices, then there are $\binom{n}{2}$ such pairs, so the latter is an upper bound for the number E of edges in the graph.■

(b) Assume that the graph is not connected, so that it splits up into a pair of nonempty graphs $(X_1, \mathcal{E}_1)$ and $(X_2, \mathcal{E}_2)$. Then for some k such that $2 \leq k \leq V-2$ the first graph has k vertices and the second has $V-k$ vertices, and therefore by two applications of (a) we have

$$E \;\leq\; \binom{k}{2} + \binom{V-k}{2} \qquad \text{which simplifies to} \qquad \binom{V}{2} - k(V-k)\;.$$

Since a graph always has at least two vertices we might as well assume that $V \geq 4$; if $V \leq 3$ one can verify this directly by looking at graphs with two or three vertices (there are only a handful of possibilities).

What is the maximum value of the displayed expression for the values of k which interest us? If $1 \leq k \leq V-1$, then the maximum values occur when $k(V-k)$ is minimized, and standard considerations about quadratic polynomials imply this happens when $k = 1$ or $V-1$. In these cases we see that

$$\binom{k}{2} + \binom{V-k}{2} \;\leq\; \binom{1}{2} + \binom{V-1}{2} \;=\; 0 + \binom{V-1}{2}\;.$$

Therefore if the graph is disconnected, so that it splits into two disjoint proper subgraphs, then $E < \binom{V-1}{2}$, and this is equivalent to saying that if $E \geq \binom{V-1}{2}$ then the graph must be connected.■

3. (a) For the first part, simply take the decomposition with a single edge and two vertices. Since there are only two vertices, the defining condition for a tree reduces to showing there is a unique reduced edge path joining them; one can check that such a path is given by the unique edge.

For the second part, consider the star shaped graph with vertices $v_0, \cdots, v_n$ and edges joining v_0 to v_j for $1 \leq j \leq n$. Then if $j > 0$ the path given by the edge v_0v_j is the unique reduced edge path joining the two vertices (such a path clearly has minimum length, and any edge path joining the vertices must end with this edge). Also if $i \neq j$ and both are positive, then an edge path joining these points must start with v_iv_0 and end with v_0v_j, so the unique reduced edge path is given by $v_iv_0 + v_0v_j$.■

If there was a reduced circuit $v_0v_1 + \cdots + v_nv_0$ such that all the edges are distinct, then one could decompose it into a pair of distinct edge paths $v_0v_1 + \cdots + v_{n-1}v_n$ and v_0v_n joining v_0 and v_n, and this contradicts the definition of a tree.■

4. (a) If $f : X \to Y$ is a homeomorphism of topological spaces and $A \subset X$, then we know that f maps A homeomorphically to $f[A]$ and $X - A$ homeomorphically to $Y - f[A]$. Therefore if A is finite with k elements and its complement has n connected components, then the same is true for $f[A]$ and $Y - f[A]$.■

(b) Define $\mathcal{S}(n,k,X)$ to be the family of all subsets $E \subset X$ such that $X - E$ has k connected components. Then by (a) we know that $E \in \mathcal{S}(n,k,X)$ if and only if $f[E] \in \mathcal{S}(n,k,Y)$. Since every subset of Y has the form $f[A]$ for some unique $A \subset X$, it follows that the two families $\mathcal{S}(n,k,X)$ and $\mathcal{S}(n,k,Y)$ have the same numbers of elements.

5. (a) Given the preceding exercise, one natural way to approach this is to compute some of the numbers $\mathbf{S}_{n,k}(X)$ defined there; this also corresponds to the usual textbook arguments showing that each space in this question is not homeomorphic to either of the others.

If $X = (0,1)$, then $S(1,1) = 0$ and $S(1,2) = \infty$.

If $X = [0,1]$, then $S(1,1) = 2$ and $S(1,2) = \infty$.

If $X = (0,1]$, then $S(1,1) = 1$ and $S(1,2) = \infty$.

Since the numbers $\mathbf{S}_{n,k}(X)$ are equal for all X in a given homeomorphism class of spaces, it follows that the three examples must lie in distinct homeomorphism types.■

(b) This is the same as (a) but the details are more complicated. The results are compiled in the table below.

Graph	$\mathbf{S}_{1,1}(X)$	$\mathbf{S}_{1,2}(X)$	$\mathbf{S}_{1,3}(X)$	$\mathbf{S}_{1,4}(X)$	$\mathbf{S}_{2,1}(X)$	$\mathbf{S}_{3,1}(X)$	$\mathbf{S}_{4,1}(X)$
4	∞	∞	1	0	∞	∞	0
6	∞	∞	0	0	∞	0	0
8	∞	1	0	0	∞	0	0
0	∞	0	0	0	0	0	0
A	∞	∞	0	0	∞	1	0
E	3	∞	1	0	3	1	0
I	2	∞	0	0	1	0	0
X	4	∞	0	1	4	6	4

If two of the spaces were homeomorphic, the corresponding entries in the rows for the two examples would be the same. However, for each pair of spaces there is a column such that the entries in that column and the two rows are unequal. Therefore no two of the examples can be homeomorphic.■

FURTHER REMARKS. One can approach the computation of $\mathbf{S}_{n,k}(X)$ systematically as follows: Given a finite set of points, one first sorts them by noting that each point is either a vertex or lies on an edge determined by a unique pair of vertices. There are only finitely many possibilities for each point, so one can list the possibilites by noting how many points lie in a set $\{v\}$ determined by a given vertex v and how many points lie in the "interior" of a given edge E. In each of these finitely many cases (there are at most $\binom{V+1}{2}$ cases if there are V vertices) there is a well defined number of connected components for the complement.

Finally, we note that it is fairly straightforward to write a computer program which will yield the numbers $\mathbf{S}_{n,k}(X)$ if we use an arbitrary graph X as input.

6. We begin with the following observation: *If X is a space which is a union of n connected subsets, then X has at most n connected components.* — This is true because $X = \cup\, A_j$ with A_j connected implies that each A_j is contained in a single connected component of X. Since X is a union of the subsets, every connected component must contain a point from at least one of the subsets A_j, and therefore the map sending $\{A_1, \cdots, A_n\}$ to the connected components of X must be onto. Since the domain has at most n elements, the same must be true of the domain, which is just the set of connected components of X.

We shall now apply this to the problem. We know that X is the union of the edges E_α, and if A is a finite subset then $X - A$ is the union of the sets $E_\alpha - A = E_\alpha - (A \cap E_\alpha)$. Since E_α is homeomorphic to $[0,1]$ and each subset of A is finite, it follows that $E_\alpha - A$ is homeomorphic to $[0,1]$ with finitely many points deleted. Such a set is equal to a union of intervals which may be open, half open or even closed (in the case where nothing is removed). Therefore each set in the finite family $\{E_\alpha - A\}$ has only finitely many connected components, and hence $X - A$ itself must be a union of finitely connected subspaces. By the comments in the first paragraph, it follows that X has only finitely many connected components.∎

V.3 : Chains, homology and fundamental groups

Additional exercises

1. One simple way of solving the first part is to use the fact that $V = E = n$ for the graph in question. By construction, the graph is connected because every vertex can be joined to v_0 by an edge path of the form $v_1v_2 + \cdots + v_{k-1}v_k$. Therefore the formulas in `notes5.pdf` show that $\dim H_1(X, \mathcal{E}^\omega; \mathbb{F}) = 1 - E + V = 1$. However, we can also find an explicit circuit because

$$v_1v_2 \ + \ \cdots \ + \ v_{n-1}v_n \ - v_1v_n$$

is a cycle (compute its boundary directly).∎

2. Since every pair of vertices lie on some edge, the graph is definitely connected. Furthermore, if there are n vertices there must be $\binom{n}{2}$ edges, so that

$$\dim H_1(X, \mathcal{E}^\omega; \mathbb{F}) \quad = \quad 1 \ + \ \binom{n}{2} \ - \ n \ .$$

Note that the right hand side is nonnegative if $n \geq 2$ and positive if $n \geq 3$.∎

3. For $i = 1, 2$ let E_i and V_i denote the number of edges and vertices in $(X_i, \mathcal{E}_i)$. If E and V denote the numbers of edges and vertices in $(X, \mathcal{E})$, then $E = E_1 + E_2$ and $V = V_1 + V_2 - 1$ because

(1) every edge in $(X, \mathcal{E})$ is contained in exactly one of the subgraphs,

(2) exactly one vertex which lies in $X_1 \cap X_2$, and each of the remaining vertices belongs to exactly one of the subgraphs.

Therefore we have

$$\dim H_1(X, \mathcal{E}^\omega; \mathbb{F}) \ = \ 1 \ + \ E \ - \ V \ = \ 1 \ + \ (E_1 + E_2) \ - \ (V_1 + V_2 - 1) \ =$$

$$2 + E_1 + E_2 - V_1 - V_2 \ = \ (1 + E_1 - V_1) + (1 + E_2 - V_2) \ = \ \dim H_1(X_1, \mathcal{E}_1^\omega; \mathbb{F}) + \dim H_1(X_2, \mathcal{E}_2^\omega; \mathbb{F})$$

which is what we wanted to prove.∎

4. We know that $\dim H_1(X, \mathcal{E}^\omega; \mathbb{F}) = 1 + E - V$, so everything reduces to computing V and E. There are $2n + 1$ vertices (one a and n each of b's and c's), and there are $3n$ edges (ab_j, ac_j and b_jc_j for $1 \leq j \leq n$). Therefore the dimension is equal to $1 + 3n - (2n + 1) = n$.∎

V.4 : Euler paths

Additional exercises

1. Express the closed Euler path in the form $x_1 x_2 + \cdots + x_{n-1} x_n + x_n x_1$ such that each edge appears exactly once. WARNING: We are not making any assumptions that the x_j are distinct or that the subscripts correspond to the linear ordering given by ω.

There is a well defined continuous mapping from the graph in Exercise V.3.1 to the graph in this exercise; specifically, take the map φ sending a vertex v_j in the first graph to a vertex x_j in the second, and extend this to each edge $v_i v_j$ such that φ is mapped homeomorphically to the edge $x_i x_j$. This map is continuous and onto, and since both the domain and codomain are compact Hausdorff spaces it follows that φ is a quotient map, where the equivalence classes are inverse images of one point sets. The only equivalence classes which contain more than one point are some of the sets $\varphi^{-1}[\{x_j\}]$ where x_j is a vertex of $(X, \mathcal{E})$. The drawing in `bowtie-graph.pdf` depicts the simplest example in which an equivalence class consist of more than one point.■

2. YES. Consider the graph with vertices a, b, c, d, x, y and the following edges:

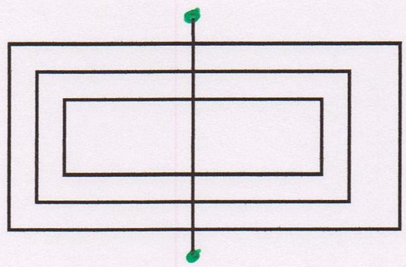
$$ab \qquad bc \qquad cd \qquad ad \qquad ax \qquad ay \qquad xy$$

Geometrically, this is a union of a triangle graph and a square graph which meet in a single vertex (namely, a).■

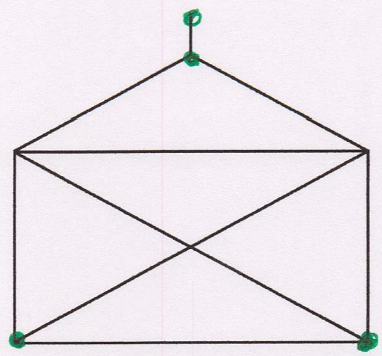
3. Examples (a), (c) and (d) have two odd vertices, Example (f) has no odd vertices, and Examples (b) and (e) have four and ten odd vertices respectively (in the drawings on the first two pages of `solutions05as15.pdf`, the odd vertices are colored green). Therefore Euler paths exist for (a), (c), (d) and (f), but they do not exist for (b) and (e). Therefore the first example with an Euler path is (a), and one such edge path is described on the third page of `solutions05as15.pdf`, with the edges numbered in the order that they appear in the path's description. The fourth page describes an Eulerpath for (c).■

Examples for Additional Exercise V.4.3

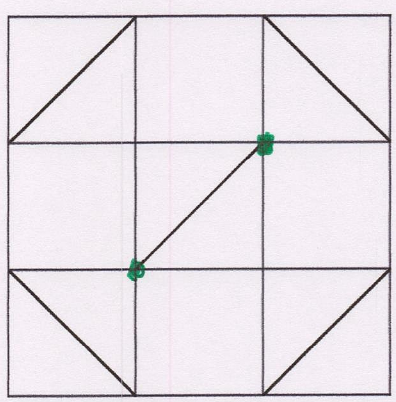
(*a*)



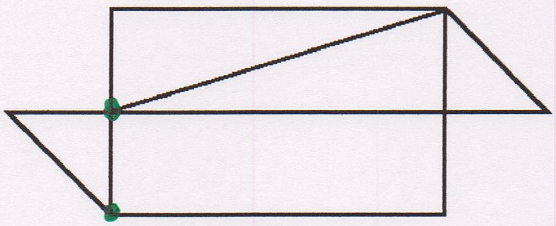
(*b*)



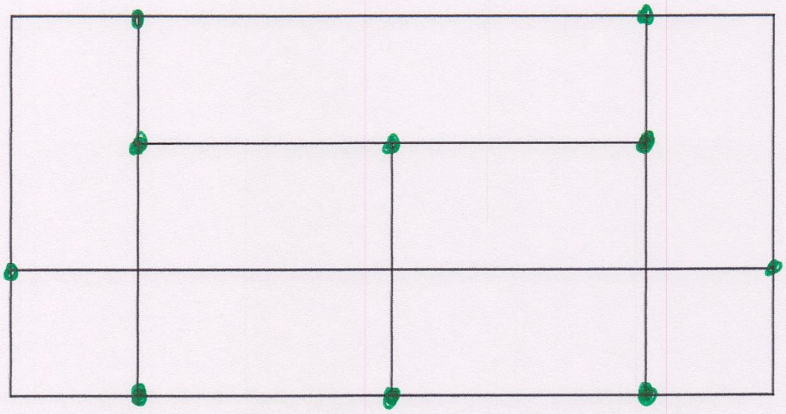
(*c*)



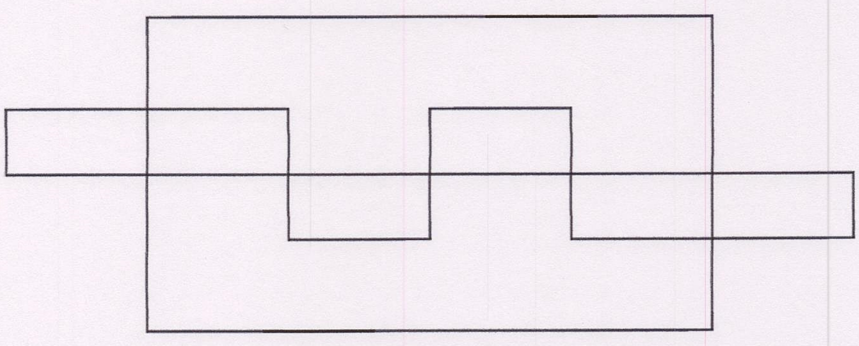
(d)

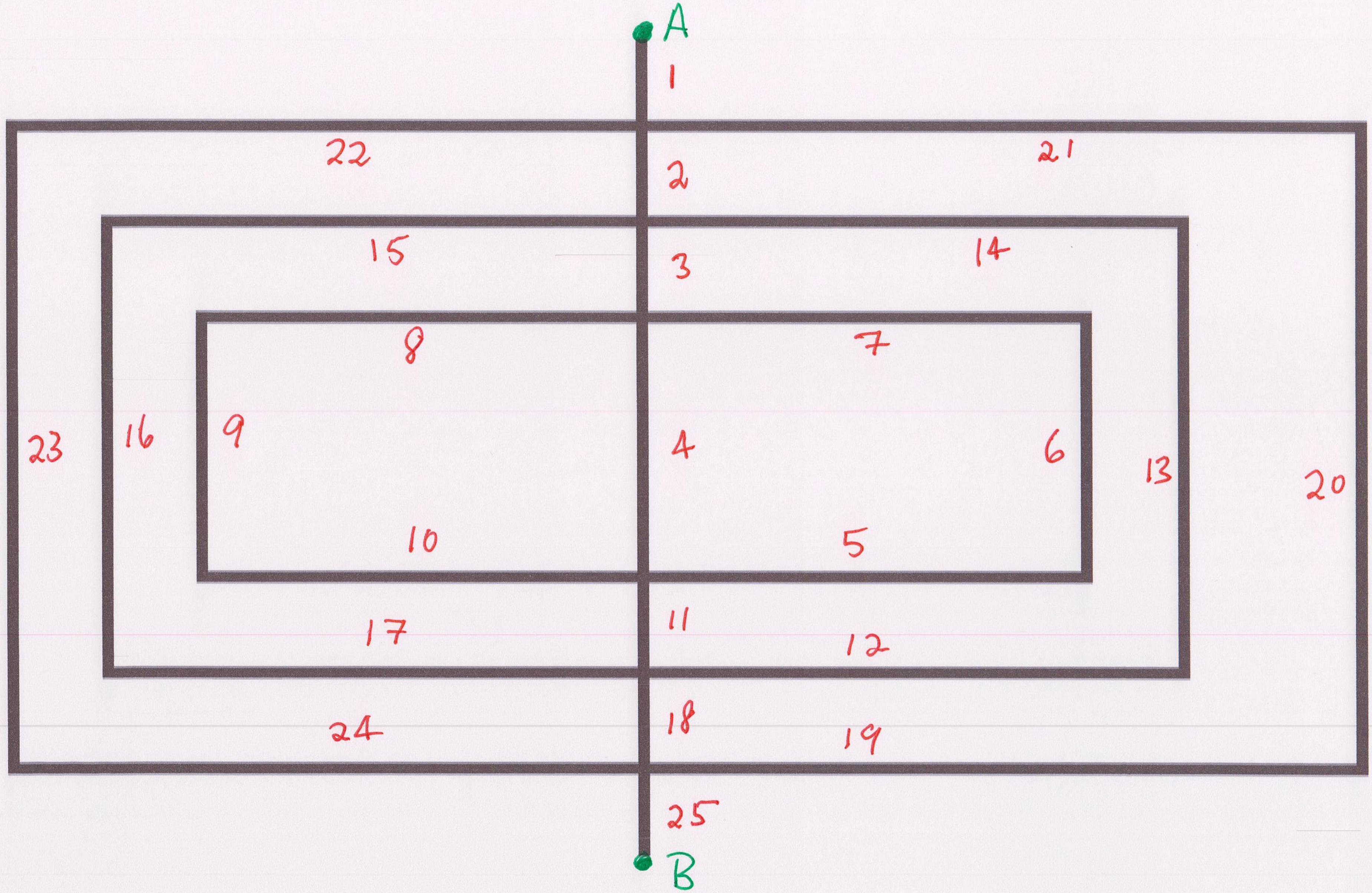


(e)



(f)

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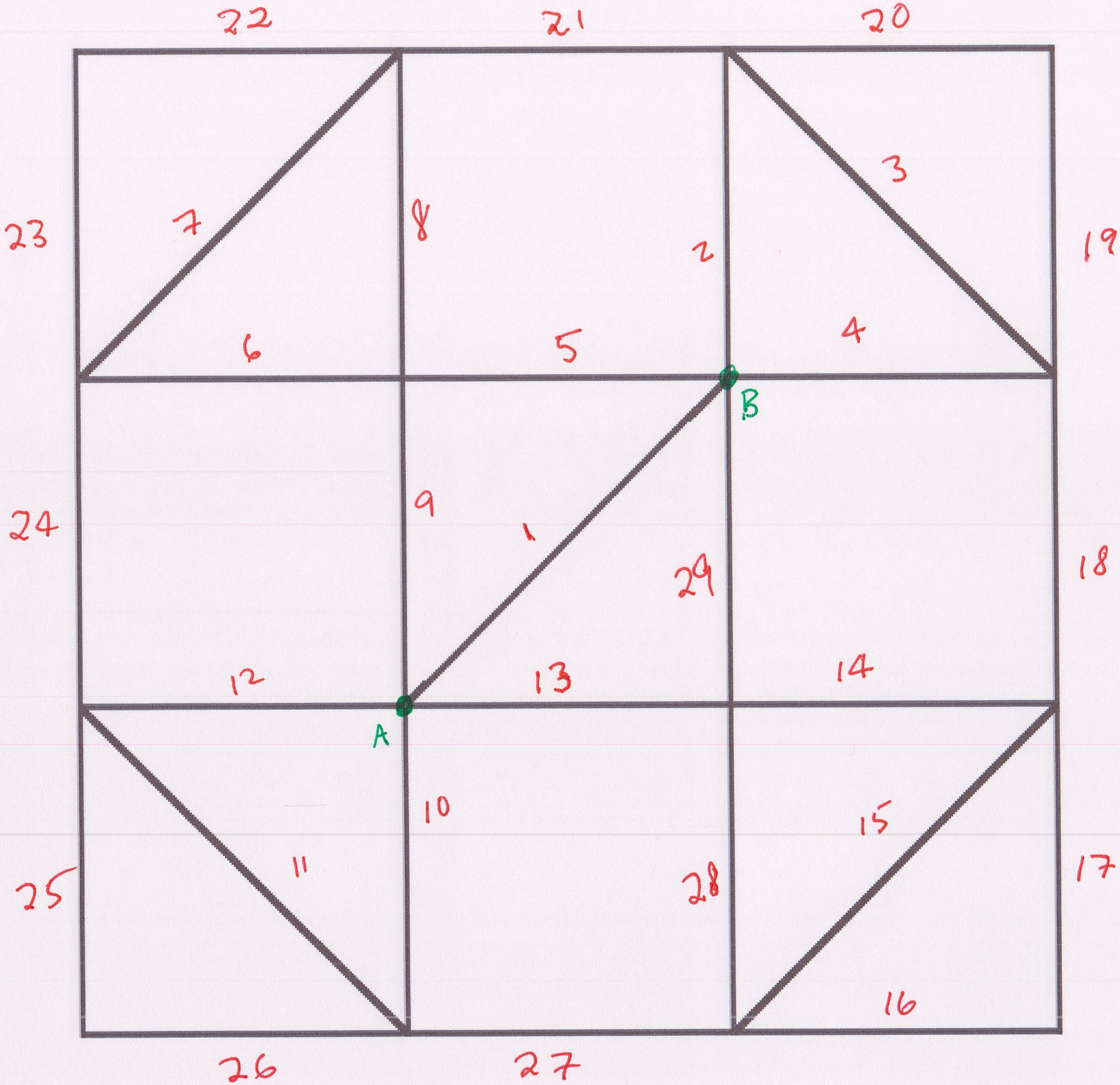
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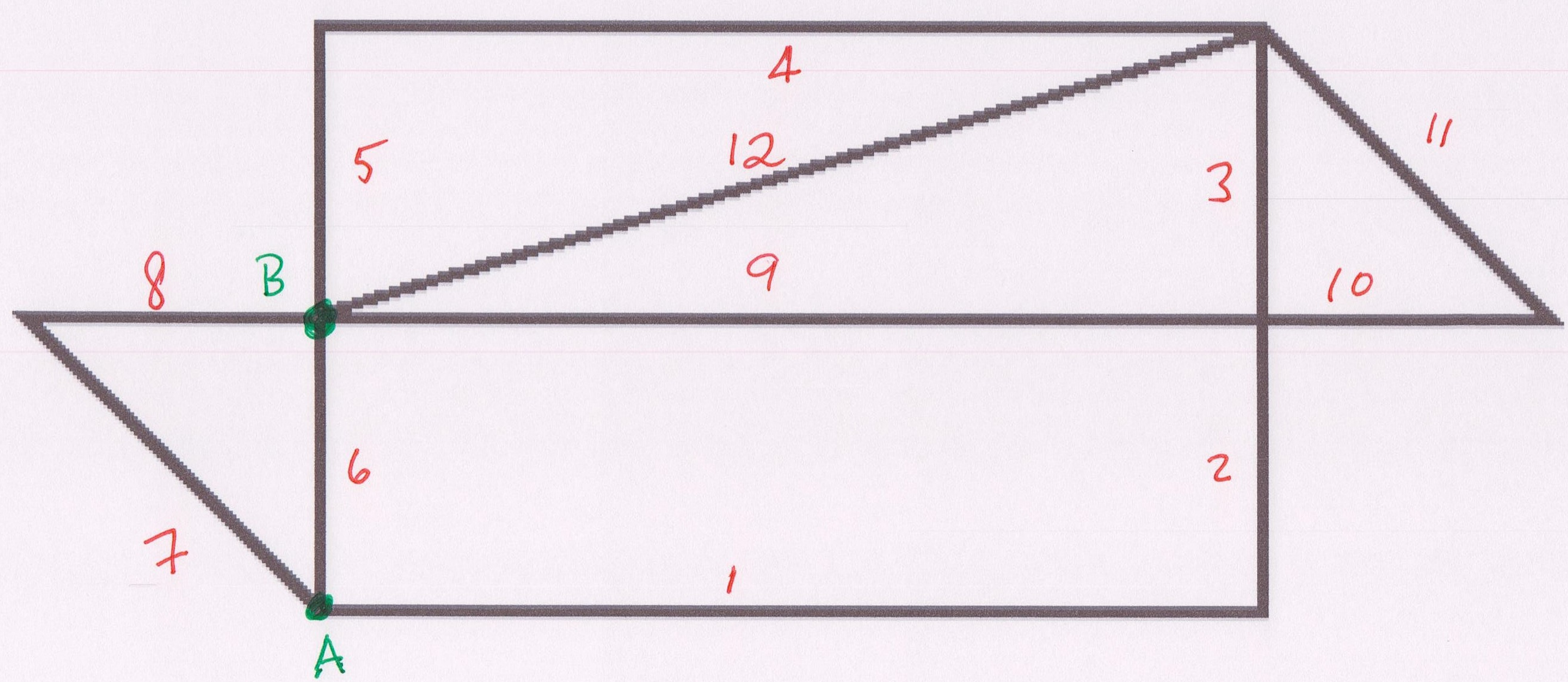
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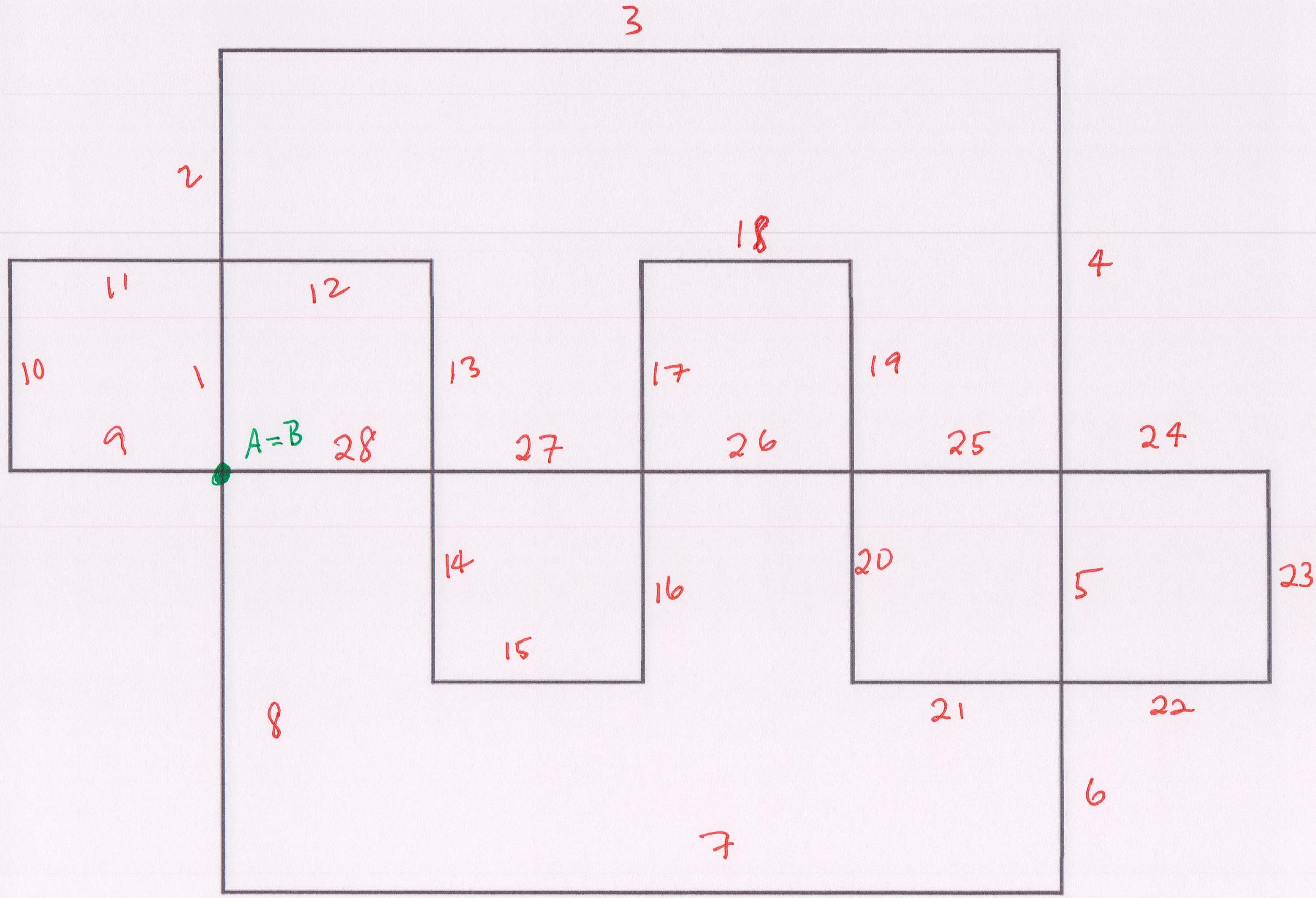
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