SOLUTION TO EXERCISE 05.10

10. (a) Let X be an arcwise connected space such that $H_q(X)$ is finitely generated and free abelian for each $q \geq 0$. Define the $\boxed{\text{Poincaré series}}$

$$P_X(t) = \sum_{q \geq 0} [\operatorname{rank} H_q(X)] \cdot t^q.$$

By our previous results on the homology of $S^n \times Y$, we know that if Y satisfies the preceding assumptions then so does $S^n \times Y$, and in fact we have $P[S^n \times Y](t) = P[Y](t) \cdot (1+t^n)$. Note also that if $H_q = 0$ for sufficiently large q, then $P_X(1)$ is defined and equals $\sum_{q \geq 0} \operatorname{rank} H_q(X)$.

If we apply this to a product of k spheres, then we get $P_X(1) = 2$ if $k=1$, and in fact we have

$$P[S^{n_1} \times \cdots \times S^{n_k}] = P[S^{n_1} \times \cdots \times S^{n_{k-1}}](1+t^{n_k}) = \cdots = \prod_{j=1}^{k} (1+t^{n_j}).$$ Hence the value at 1 is 2^k and the latter is the sum of the ranks of $H_q(S^{n_1} \times \cdots \times S^{n_k})$. ∎

(b) By (a), if the two products have isomorphic homology groups, then $2^p = 2^q$, which implies that $p = q = k$ (CALL IT THIS)

The conclusion of the exercise is trivial if $k=1$, so assume it is known for $(k-1)$-fold products. Suppose that $H_*(S^{n_1} \times \dots \times S^{n_k}) \cong H_*(S^{m_1} \times \dots \times S^{m_k})$.

Consider the first $d > 0$ such that $H_d \neq 0$. For the first space we have $d = n_1$, while for the second we have $d = m_1$, so $n_1 = m_1$. We then have

$$P(S^{n_2} \times \dots \times S^{n_k}) = \frac{P(S^{n_1} \times \dots \times S^{n_k})}{(1+t^{n_1})} \underset{\substack{\text{hypotheses} \\ \text{and} \\ m_1 = n_1}}{\overset{\text{by}}{=}}$$

$$\frac{P(S^{m_1} \times \dots \times S^{m_k})}{(1+t^{m_1})} = P(S^{m_2} \times \dots \times S^{m_k}).$$

Now the induction hypotheses can be applied to see that $n_j = m_j$ for $2 \leq j \leq k$. ∎