Math 205B
Winter 2018

UPDATED GENERAL INFORMATION — MARCH 18, 2018

Solutions to practice problems from `review2.pdf`

1. Follow the hint. The composite mapping $j \circ g : U \to S^m$ is continuous and 1–1, so by Invariance of Domain it is also an open mapping. On the other hand, its image is contained in $S^n \subset S^m$ (the standard embedding such that the last $m-n$ coordinates are zero), and no nonempty subset $A \subset S^n$ is open in S^m. There are several related ways of showing the latter. For example, if A were open with $a \in A$, then some open neighborhood $N_\varepsilon(a)$ of radius ε is contained in A; however, this cannot happen, for we can always find a sequence of points $\{x_n\}$ in $S^m - S^n \subset S^n - A$ whose limit is a.■

2. Again, follow the hint and start by showing the intersection is starshaped. Since $\mathbf{K}_0 \cap \mathbf{A}$ consists of all maximal faces except one, there is a unique vertex v of $\mathbf{A}$ which is not a vertex of the excluded face. This vertex must then be a vertex of all the remaining maximal faces, and hence $\mathbf{K}_0 \cap \mathbf{A}$ must be starshaped with respect to v. Therefore $H_q(\mathbf{K}_0 \cap \mathbf{A}) = 0$ if $q \neq 0$ and $H_0(\mathbf{K}_0 \cap \mathbf{A}) = \mathbb{Z}$. This implies that $H_q(\mathbf{A}, \mathbf{K}_0 \cap \mathbf{A}) = 0$ for all q. By Simplicial Excision we know that $H_q(\mathbf{A}, \mathbf{K}_0 \cap \mathbf{A}) \cong H_q(\mathbf{K}, \mathbf{K}_0)$, so the latter groups are also trivial. Finally, we can apply the long exact homology sequence of the larger pair to conclude that the map $H_q(\mathbf{K}_0) \to H_q(\mathbf{K})$ is an isomorphism for all q.■

3. In order to compute the homology groups of the threefold union, we must first compute the homology groups of the twofold unions $U \cup V$, $U \cup W$ and $V \cup W$. **At this point there should be an additional assumption that each of the twofold intersections is nonempty.**

We claim that the reduced homology groups of each of the twofold unions are zero in all dimensions. It will suffice to prove this for $U \cup V$, for the remaining cases will follow by changing the variables in the argument. The intersection of the two convex open subsets U and V is convex, and therefore in the Mayer-Vietoris sequence relating the reduced homologies of $U \cap V$, U, V and $U \cup V$ the terms $\widetilde{H_q}(U \cap V)$ and $\widetilde{H_q}(U) \oplus \widetilde{H_q}(V)$ are all zero. Therefore the third terms in the exact Mayer-Vietoris sequence, which are the groups $\widetilde{H_q}(U \cup V)$, must also be zero.

The next step is to compute the homology of the threefold union $U \cap V \cap W$, viewing it as the union of $U \cap V$ and W. By the preceding paragraph we know that the reduced homology groups of both $U \cap V$ and W are trivial. The intersection of these sets is given by $(U \cap W) \cup (V \cap W)$, and this is a union of two convex sets. If the sets $(U \cap W)$ and $(V \cap W)$ are disjoint, then the reduced homology of their intersection is $\mathbb{Z}$ in dimension zero and 0 otherwise, and by the preceding paragraph the reduced homlogy of their union is 0 in all dimensions if $(U \cap W)$ and $(V \cap W)$ are not disjoint. In either case we know that the homology of the intersection vanishes in positive dimensions. Now the exact Mayer-Vietoris sequence for $\widetilde{H_q}(U \cup V \cup W)$ shows this group is isomorphic to $\widetilde{H_{q-1}}(U \cap V \cap W)$, and since the latter is 0 if $q - 1 > 0$ it follows that the former is 0 if $q \geq 2$. Since reduced homology agrees with homology in positive dimensions, it follows that $H_q(U \cup V \cup W) = 0$ if $q \geq 2$.

Finally, we need an example where $H_q(U \cup V \cup W) = \mathbb{Z}$. The hint suggests thickening up the three edges of a triangular graph, and the file `review2fig1.pdf` illustrates how this can be done.■

4. Since f_* is invertible, the integral 3×3 matrix A is also invertible. By Cramer's Rule an integral matrix has an integral inverse if and only if $\det A = \pm 1$.■

5. The mappings i_{1*} and i_{2*} are 1–1 because the slice inclusions i_1 and i_2 are retracts. Specifically, one-sided inverses are given by the projections onto the first and second coordinates respectively. We need to show that their intersection is the zero subgroup.

Suppose we are given $w \in H_1(X \times Y)$ such that $i_{1*}(u) = i_{2*}(v)$ for suitable u and v. The mapping $\pi_Y \circ i_1$ sends every $x \in X$ to the same $q \in Y$, so it is constant and the induced mapping in homology $\pi_{Y*} \circ i_{1*}$ is also trivial. Therefore $0 = \pi_{Y*} \circ i_{1*}(u) = \pi_{Y*} \circ i_{2*}(v) = v$, which means that $w = i_{2*}(v) = i_{2*}(0) = 0$.■

6. The homology groups of C_* and $(D/C)_*$ are equal to the chain groups because the boundary (or differential) maps in the chain complexes are all zero. Suppose now that we are given a cycle in $H_{q+1}(D/C) \cong (D/C)_{q+1} = A$; call it a. The first step in constructing ∂ is to lift this cycle to an element of D_{q+1}, which is also A. The map from D_{q+1} to $(D/C)_{q+1} = A$ is the identity, so $a \in D_{q+1} = A$ is an obvious choice for a lifting. Now apply the differential in the chain complex D_*. This is given by $d : A \to B$, and therefore we obtain the class $d(a) \in D_q = B$. The latter has a unique lifting to $C_q = B$. Therefore the mapping ∂ sends a to $d(a)$ (there is an obvious misprint in `review2.pdf`, in which f should be replaced by d).■

7. For the algebraic part, let $i : A \to B$ and $j : B \to X$ be the inclusions. Then i and $j \circ i$ are both homotopy equivalences and hence induce isomorphisms in homology. We need to show that j_* is also an isomorphism in homology.

By functoriality we know that $(j \circ i)_* = j_* \circ i_*$, and we know that this and i_* induce isomorphisms in homology. The mapping j_* must be onto because $w \in H_q(X)$ implies $w = (j \circ i)_*(u) = j_*\,(i_*(u)\,)$ for some $u \in H_q(A)$. To see that j_* is 1–1, suppose that $j_*(v) = 0$. Since i_* is onto we have $v = i_*(u)$ for some $u \in H_q(A)$, and clearly $0 = j_*(v) == j_*\,(i_*(u)\,) = (j \circ i)_*(u)$. But $(j \circ i)_*$ is an isomorphism, so it follows that $u = 0$ and hence also that $0 = i_*(u) = v$.

Now for the counterexample. Let $X = D^n$, $A = \{\mathbf{0}\}$ and $B = D^n - S^{n-1}$. Then A is a strong deformation retract of both B and X. However, B is not a strong deformation retract of X. If so, there would be a continuous mapping $\rho : X \to B$ such that $\rho|B$ is the identity. Consider the mapping $i \circ \rho$, where $i : B \to X$ is inclusion. By construction, the restriction of this map to B is equal to the i, which is also the restriction of the id_X to B. Since X is Hausdorff, the set of points where $\rho = \mathbf{id}_X$ is closed in X. Now B is dense in X, so this means that $\rho = \mathbf{id}_X$; but the map on the left is not onto while the map on the right is onto, so we have a contradiction. The source of this contradiction is the assumption that a map like ρ exists, and therefore there is no such mapping; in other words, B is not a retract of X.■

8. By construction X_1 is a graph with six edges which all meet at a common vertex. This means that X_1 has the homology of a point. Furthermore, $X_0 \cap X_1$ consists of the "other" endpoints of the edges, and hence it consists of six points. We can then use the simplicial Mayer-Vietoris sequence to compute the homology of $X = X_1 \cup X_2$ as follows; since $H_q(X) = 0$ if $q \geq 3$ and $\dim X \leq 2$, we need only consider the tail end:

$$H_2(X_0 \cap X_1) \longrightarrow H_2(X_0) \oplus H_2(X_1) \longrightarrow H_2(X) \longrightarrow$$

$$H_1(X_0 \cap X_1) \longrightarrow H_1(X_0) \oplus H_1(X_1) \longrightarrow H_1(X) \longrightarrow$$

$$H_0(X_0 \cap X_1) \longrightarrow H_0(X_0) \oplus H_0(X_1) \longrightarrow H_0(X)$$

Since X is arcwise connected, the last group is $\mathbb{Z}$. In the remaining cases, substitution of known groups yields the following:

$$0 \to \mathbb{Z} \oplus 0 \to H_2 \to 0 \to 0 \oplus 0 \to H_1(X) \to \mathbb{Z}^6 \to \mathbb{Z} \oplus \mathbb{Z}$$

Furthermore, the last map in this sequence sends the 6-tuple $(x_1, ...x_6)$ to $(s, -s)$ where $s = \sum x_i$. This exact sequence implies that $H_2 \cong \mathbb{Z}$ and $H_1 \cong \mathbb{Z}^5$.■

9. The key property of S^2 and $T^2 = S^1 \times S^1$ is that they are surfaces: Both are Hausdorff, and each point of the space has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^2$. One can then extend Invariance of Domain to a result about surfaces: *If X and Y are surfaces and $f : X \to Y$ is continuous, then f is an open mapping.*

It suffices to prove this for all open sets in an open covering of X. Start with an open covering of Y by open subsets V_α homeomorphic to open subsets of $\mathbb{R}^2$, and construct an open covering of X by open subsets U_β such that f maps each U_β into some V_α. Then Invariance of Domain implies that each $f|U_\beta$ is an open mapping, and from this it follows that f itself must be an open mapping.

Using this, proceed as follows: Let X and Y be S^2 and T^2 respectively or vice versa. Then by the preceding paragraph f is open and hence $f[X]$ is open in Y. But X and Y are compact and connected. Therefore $f[X]$ is also compact, and hence closed in Y. Since Y is connected and $f[X]$ is a nonempty open and closed subset, we must have $f[X] = Y$ and f is a homeomorphism. Since X and Y are not homeomorphic, this is a contradiction and hence no such mapping f can exist.■

10. Let $p : (E, e) \to (B, b)$ be the associated basepoint preserving covering space projection. Then the induced map of fundamental groups $p_* : \pi_1(E, e) \to \pi_1(B, b)$ is the trivial homomorphism by our hypothesis. Since p_* is always 1–1, it follows that $\pi_1(E, e)$ must be the trivial group. Since E is assumed to be arcwise connected, it follows that E must also be simply connected.■