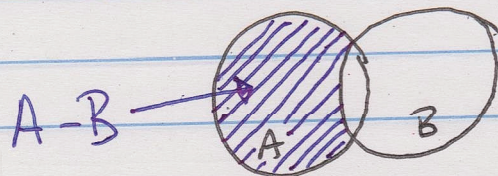


2. NOTATION & TERMINOLOGY

A course in set theory is a prerequisite.
 Set theoretic notation and vocabulary
 will be used freely without explanations.
 (This includes unions, intersections, ...)

Notation for relative complements: $A - B$
 (Sutherland uses $A \setminus B$).



Standard notation for intervals in reals (\mathbb{R})

$[a, b]$	(a, b)	$[a, b)$	$(a, b]$
CLOSED	OPEN		
CLOSED		HALF-OPEN	

At the open ends we allow values of $\pm\infty$.

Functions (maps, mappings) (or morphisms)

Formally, these are triples

$f = (A, \Gamma_f, B)$ $\Gamma_f \subseteq A \times B$ such that

for each $a \in A$ there is a unique $b \in B$ such that
 $(a, b) \in \Gamma_f$. Write $b = f(a)$ if $(a, b) \in \Gamma_f$.

Note that $A \times B =$ all ordered pairs (a, b) such that $a \in A$ & $b \in B$.

Given two objects a & b one can form an ordered pair (a, b) such that $(a, b) = (c, d) \Leftrightarrow a = c$ & $b = d$

Write $f: A \rightarrow B$ and say that A is the domain of f , B is the co-domain of f .
(Not everyone specifies codomains explicitly,

but ultimately this is necessary in many contexts. — There are analogies in computer languages, where one must specify whether the values of certain functions are whole numbers or decimal expressions.)

Synonyms for special types of functions

1-1 into
injective
monomorphism

onto
surjective
epimorphism

(BOTH VALID)

1-1 onto or 1-1 correspondence
bijective
isomorphism

3. MORE ON SETS & FUNCTIONS

Main concepts $f: A \rightarrow B$ function
also sometimes $A \xrightarrow{f} B$

(1) If $C \subseteq A$, the image of C under f :

(2) If $D \subseteq B$, the inverse image of D with respect to f .

(3) Inverse functions.

All play ~~an~~ important roles in the course.

(1) Images

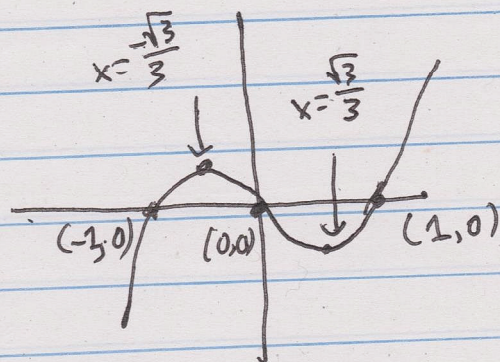
$f[C] =$ all $b \in B$ such that $b = f(c)$,
some $c \in C$.

(2) $f^{-1}[D] =$ all $a \in A$ such that $f(a) \in D$.

Two examples are on p. 10 of Sutherland.

Here is one more.

$$f(x) = x^3 - x.$$



$$f: \mathbb{R} \rightarrow \mathbb{R}$$

Suppose $C = [0, 1]$. Then

$$f[C] = \left[\left(\frac{\sqrt{3}}{3}\right)^3 - \left(\frac{\sqrt{3}}{3}\right), 0 \right].$$

{NO PROOF!}

↑
minimum value.

Suppose $D = (-\infty, 0]$.

$$\text{Then } f^{-1}[D] = (-\infty, -1] \cup [0, 1].$$

Yet another example

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = \frac{1}{x^2 + 1}.$$

$$f[\mathbb{R}] = (0, 1]$$

(since $0 < f(x) < 1$)

$$f^{-1}[D] = \mathbb{R} \text{ if } D = [0, 1] \text{ or } [0, 2]$$

$$f^{-1}[D] = \emptyset \text{ if } D = [-1, 0].$$

$$\text{If } C = [0, 1], \text{ then } f[C] = \left[\frac{1}{2}, 1\right].$$

Variant identities are on pp. 10-13 of

Sutherland. For example

$$f[C_1 \cup C_2] = f[C_1] \cup f[C_2]$$

$$f^{-1}[D_1 \cup D_2] = f^{-1}[D_1] \cup f^{-1}[D_2]$$

$$f^{-1}[D_1 \cap D_2] = f^{-1}[D_1] \cap f^{-1}[D_2].$$

However, we only have

$$f[C_1 \cap C_2] \subseteq f[C_1] \cap f[C_2].$$

Example $C_1 = [-2, -1]$ $C_2 = [1, 2]$

$$f(x) = x^2. \quad C_1 \cap C_2 = \emptyset \Rightarrow \text{LHS} = \emptyset$$

$$\text{but RHS} = [1, 4].$$

Potential source of confusion

The terminology $f^{-1}[D]$ does not mean that there is an inverse function f^{-1} . !!

Two characterizations of functions with inverses (invertible functions) — or is it three?

An inverse function $g: Y \rightarrow X$ to $f: X \rightarrow Y$ has the property that $x = g(y) \iff y = f(x)$.

(1) If f is 1-1 onto, then one has such a function because for each $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$.

IMPORTANT !! (2) Suppose that there is $h: Y \rightarrow X$ such that $h(f(x)) = x$ all x and $f(h(y)) = y$ all y . Then f is 1-1 onto and h is an inverse.

Proof of (2) f is 1-1 $f(x) = f(x') \Rightarrow$

$$x = h(f(x)) = h(f(x')) = x'$$

f is onto $y = f(h(y))$ for all $y \in Y$.

inverse identities $y = f(x) \Rightarrow x = h(f(x)) = h(y)$.

$$x = h(y) \Rightarrow f(x) = f(h(y)) = y. \quad \blacksquare$$

BIG DOT =
END OF
ARGUMENT.

We often, but not always,
write $h = f^{-1}$

Sutherland, Proposition 3.20, rewritten

$f: X \rightarrow Y$ 1-1 onto, $h =$ inverse for.

$V \subseteq X$. Then $f[V] = h^{-1}[V]$.

Proof. $y = f(v)$ for some $v \in V \Leftrightarrow$

$v = h(y)$. The (\Rightarrow) implication

shows that $f[V] \subseteq h^{-1}[V]$, and the

(\Leftarrow) implication shows that $h^{-1}[V] \subseteq f[V]$. \blacksquare