Some informal references for universal mapping properties

A major difference between graduate and undergraduate mathematics is the frequent use of ***Universal Mapping Properties*** to characterize various constructions at higher levels of the subject. In many cases these properties yield most of the information needed to work with the constructions, just like the axiomatic characterization of the real numbers (a complete ordered field) allows one to use real numbers without getting distracted by the details of the explicit construction.

The ***Wikipedia*** article on free groups

http://en.wikipedia.org/wiki/Free_group

is a good place to start (comments on the reliability of such articles have been made in another document; the citations of such references in this document can be viewed as confirmation on their reliability). However, the section on Tarski's problems should be skipped. One suggestion is to understand the article so that you are extremely comfortable with this notion first and then to consider free products. In the article

http://en.wikipedia.org/wiki/Free_product

Looking at the first three paragraphs is the first recommendation, after which one should skip to the first three sections under "contents." Again, you should try to make yourself as comfortable as possible with the contents. The discussion of free groups in the "Examples" section of

http://en.wikipedia.org/wiki/Free_object

might also be helpful.

Finally, we should note that Universal Mapping Properties are implicit in some standard constructions in undergraduate mathematics. The ring of polynomials over a ring is one basic example. In this case the Universal Mapping Property is given by the following result:

Theorem. *Let* **R** *be a commutative ring with unit, let* **P** *be a commutative ring with unit containing* **R** *(with the same unit), and let* **S** *be a subset of* **P.** *Then* **P** *is isomorphic to the polynomial ring* **R[S]** *(such that the copies of* **R** *and* **S** *correspond in the obvious manner) if and only if one has the following Universal Mapping Property*:

> *If* **A** *is a commutative ring with unit and we are given* **(1)** *a unit preserving ring homomorphism* $\mathbf{h : R \longrightarrow P}$, **(2)** *a set – theoretic map* **g** *from* **S** *to* **P,** *then there is a unique unit preserving ring homomorphism* $\mathbf{h' : R[S] \longrightarrow P}$ *which agrees with* **h** *on* **R** *and with* **g** *on* **S.**

The proof of this result is left to the reader.

Although the preceding indicates that Universal Mapping Properties are implicit in some part of undergraduate mathematics, it is at the beginning graduate level that they become extremely useful and worth the trouble of formulating them abstractly. Often it takes some time and experience to get used to them, but it is worth the effort.