Covering Space Transformations

$p : (X, x_0) \longrightarrow (Y, y_0)$ covering space projection

(X & Y conn., loc. arc. conn., T_2)

Automorphisms (deck transformations) of p = maps $T : X \longrightarrow X$ s.t.

$p \circ T = p$

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

Note: T need not preserve basepoints and usually doesn't

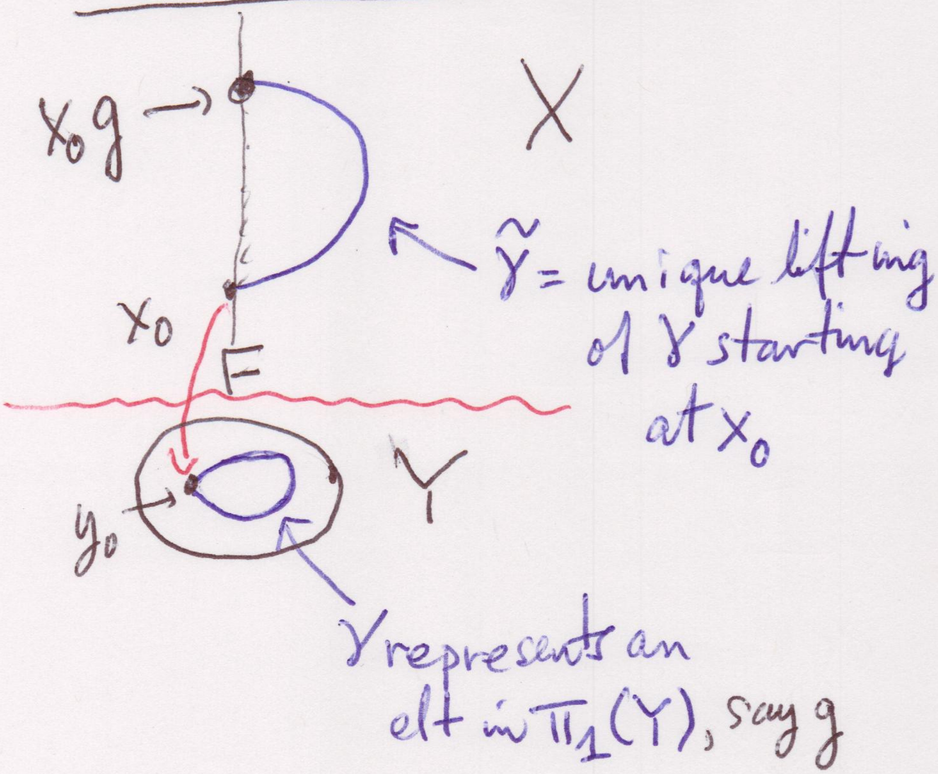
We want to describe these completely.

First step. $F = p^{-1}[\{y_0\}] = \boxed{\text{fiber}}$ of basepoint.

CLAIM $\pi_1(Y, y_0)$ permutes F transitively.

The stabilizer of $x_0 \in F$ is $p_*[\pi_1(X, x_0)]$.

<u>Construction</u>



Need to check that

① $x_0 g$ only depends on class of γ in π_1

② $(x_0 g_1) g_2 = x_0 (g_1 g_2)$

③ $x_0 g = x_0 \iff g \in \mathrm{Im}\, p_*$.

④ if $z \in F$, then $z = x_0 g$ for some g (join z to x_0; the projection of this curve on Y is a closed curve representing some $g \in \pi_1(X, x_0)$, and $z = x_0 g$.

The preceding gives a geometric way of interpreting $\pi_1(Y, y_0)/\text{Image } p_*$.

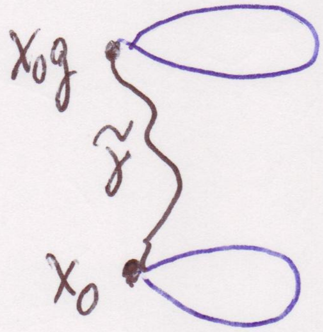
<u>Prop.</u> If T and T' are covering space projections s.t. $T(x_0) = T'(x_0)$, then $T = T'$.

<u>Hint:</u> Show the set of all points where $T = T'$ is open; since it's also closed and not empty, it must be all of X by connectedness.

<u>Question</u> If $g \in \pi_1(Y, y_0)$, when does the map $F \longrightarrow F$, sending x to xg, extend to all of X continuously (as a covering space transformation)?

Suppose there is some $T : X \to X$ s.t.

$T(x_0) = x_0 g$.



$\tilde{\alpha}$ ~~[illegible]~~ must be $T \circ \alpha$

$\tilde{\gamma} + \tilde{\alpha} + (-\tilde{\gamma})$ is a lifting of/for gag^{-1}, where

α = any closed curve based at x

γ reps $g \in \pi_1(Y, y_0)$, α reps $a \in \pi_1(X, x_0)$.

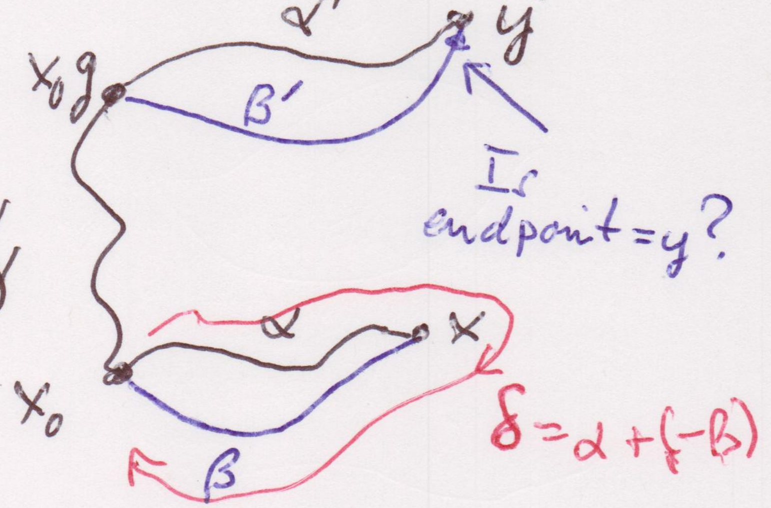
Since $\tilde{\gamma} + \tilde{\alpha} + (-\tilde{\gamma})$ is closed, $gag^{-1} \in$ Image p_*. Hence

$g \in$ Normalizer of $p_* \pi_1(X, x_0)$ in $\pi_1(Y, y_0)$

So we have an injection

$$\text{Covering transfs}\,(p : X \to Y) \longrightarrow \frac{\text{Normalizer}\ \pi_1(X)}{\pi_1(X)}.$$

Conversely, suppose $g \in$ Normalizer.



We want to define T so that $T(x_0) = x_0 g$.

Idea of construction: $x \in X$. Join x_0 to x by a curve, say α. Let α' be the unique lifting of $p \circ \alpha$ to X such that $\alpha'(0) = x_0 g$, and try to take $T(y) = \alpha'(1)$.

Is this even well-defined?

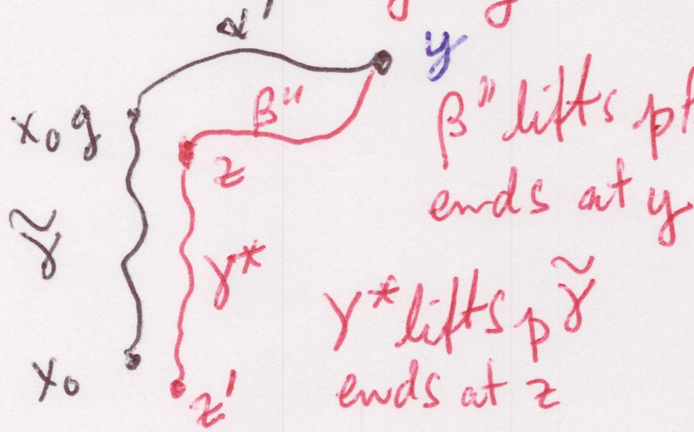
Suppose we take a second curve β joining x_0 to x and form β' as before. Is $\beta'(1) = \alpha'(1)$? [THEN need to verify the map is continous]

Let $\delta = \alpha + (-\beta)$, and let

$\Delta \in \pi_1(X, x_0) \xrightarrow{\subseteq} \pi_1(Y, y_0)$ represent it.

Since $g \in$ Normalizer, $g\Delta g^{-1} \in$ image $\pi_1(X, x_0)$. — Look at the lifting of $_p\tilde{\gamma} + p\delta + (-{}_p\tilde{\gamma})$ to X: starting at x_0

represents $g\Delta g^{-1}$



Footnote: There is a version of the Path Lifting Property in which one is given the final (not initial) value. — WHY?

Since $g\Delta g^{-1} \in$ image $\pi_1(X, x_0)$, the lifting is a closed curve, and $\boxed{x_0 = z'}$

Uniqueness of liftings now implies

$\tilde{\gamma} = \gamma^*$ and hence $z = x_0 g$.

Hence β' and β'' are liftings of $p\beta$ starting at $z = x_0 g$, so that $\beta' = \beta''$.

Since $\beta''(1) = y$, we have $\beta'(1) = y$.

Continuity Given $x \in X$ find an evenly covered neighborhood U of $p(x)$. arcwise conn

CLAIM T maps U_x to U_y. (which $\Longrightarrow$ continuity)

Let $z \in U_x$ and join it to x by a curve θ in U_x. $\theta' =$ lift of $p\theta$ starting at y.

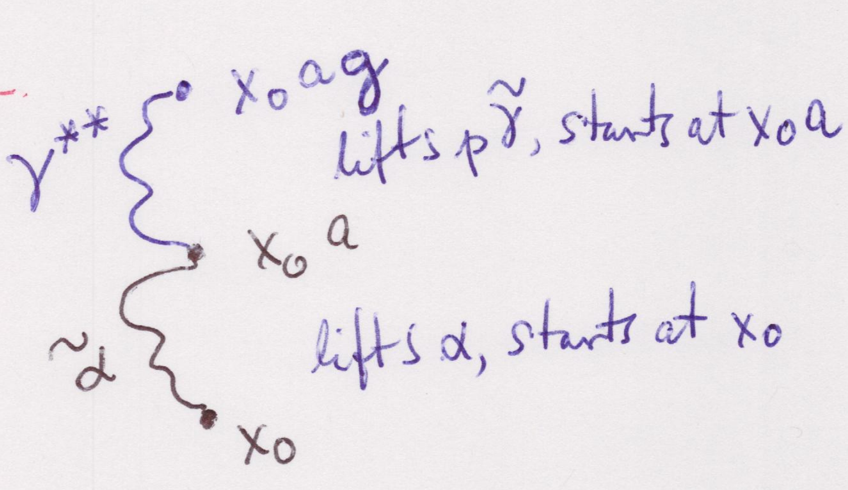
Want $z' = T(z)$. But z' is the final point of $\alpha' + \theta'$, which lifts $p\alpha + p\theta$ and starts at $x_0 g$, so this is true.

Why is T a homeomorphism?

Given $g \in$ Normalizer, let T_g be given as above.

CLAIM: If $a \in \pi_1(Y, y_0)$, then

$$T_g(x_0 a) = x_0(ag)$$



This is just the def. of $T_g(x_0 a)$, and it equals $x_0 ag$

So we have

$$T_g \circ T_a = T_{ag}$$

almost a homomorphism, but the order of multiplication is reversed [like matrix transposition]

True at x_0 by above, hence true everywhere by earlier observations.

Now $T_1 =$ identity, so this means

$$T_g \circ T_{g^{-1}} = Id = T_{g^{-1}} \circ T_g$$

which shows that $T_{g^{-1}}$ is a continuous inverse to T_g.

<u>Corollary</u> The group of covering transformations for $p: X \to Y$ is <u>transitive</u>* on $F = p^{-1}[\{y_0\}]$ $\iff$ Image $\pi_1(X, x_0)$ is normal in $\pi_1(Y, y_0)$.

*[Given $x_1 \in F$, we have $x_1 = T(x_0)$ for some cov trans T]