Exercise on orbit spaces

We shall prove the portion of Exercise 31.8 in Munkres (see p. 199) which is needed in this course. A reference for the entire proof appears below. More specifically, here is what we shall prove:

CLAIM. *If the finite group G acts on the Hausdorff space X, then the orbit space (or quotient space) X/G is also Hausdorff.*

Note that the claim contains no assumption regarding the freeness of the group action. The main step in the proof is the following strengthening of the defining property for Hausdorff spaces.

LEMMA. *Suppose that W is a Hausdorff space and E and F are disjoint finite subsets of W. then there are disjoint open subsets U and V such that $E \subset U$ and $F \subset V$.*

Proof of Lemma. Let $x_i \in E$ and $y_j \in F$. Then there are disjoint open subsets $U_{i,j}$ and $V_{i,j}$ of W such that $x_i \in U_{i,j}$ and $y_j \in V_{i,j}$. Let $M_i = \cap_j U_{i,j}$ and $N_j = \cap_i V_{i,j}$, and take $U = \cup_i M_i$ and $V = \cup_j N_j$.■

Proof of Claim. Since p is onto, we can write the two distinct points of X/G as $p(x)$ and $p(y)$ for some $x, y \in X$. By choice, the finite (hence closed) subsets $G \cdot \{x\}$ and $G \cdot \{y\}$ are disjoint. Let U_0 and V_0 be disjoint open subsets containing them, and let

$$U = \bigcap_{g \in G} g \cdot U_0 , \qquad V = \bigcap_{g \in G} g \cdot U_0 .$$

Then $G \cdot U = U$, $G \cdot V = V$ and $U \cap V = \emptyset$. By Construction $G \cdot \{x\} \subset U$ and $G \cdot \{y\} \subset V$, so that $p(x) \in p[U]$, $p(y) \in p[V]$, and $p[U] \cap p[V] = \emptyset$ (if the intersection were nonempty, then $G \cdot U$ and $G \cdot V$ would have a point in common). If $p[U]$ and $p[V]$ are open subsets, then they are the desired disjoint open neighborhoods of $p(x)$ and $p(y)$.

To conclude the argument, we shall show that if $p : X \to X/G$ is an arbitrary quotient projection (with no assumptions on G or X), then p is open. let U be open in X; then

$$p^{-1}\big[\, p[U] \,\big] = \bigcup_{g \in G} g \cdot U$$

is open, and by the defining property of the quotient topology this means that $p[U]$ is open. As noted before, this completes the proof of the claim.■

Proofs for some other parts of the exercise (and more) can be found in Theorem 3.1 on page 38 of Bredon, *Introduction to Compact Transformation Groups*. Incidentally, the conjecture stated on that page is true by work of R. Oliver.

REFERENCES

G. E. Bredon. *Introduction to Compact Transformation Groups*, Pure and Applied Mathematics Vol. 46. Academic Press, New York, 1972.

R. Oliver. *A proof of the Conner Conjecture.* Ann. Math. **103** (1976), 637–644.