Convex bodies and radial projection

Recall that a convex set in $\mathbb{R}^n$ is a set K such that if $x, y \in K$ and $0 \leq t \leq 1$ then $tx+(1-t)y \in K$. Geometrically, this means that if x and y are in K then the closed line segment joining them is also contained in K. Visually, this means that the set has no dents or holes.

Definition. A *convex body* in $\mathbb{R}^n$ is a compact convex set K with nonempty interior, and it is *regular* if it is the set of points satisfying an inequality of the form $h(v) \geq 0$ for some continuous real valued function h defined on an open neighborhood of K and the point set theoretic boundary (or frontier) ∂K of K is the set of all points where $h(v) = 0$.

DEFAULT HYPOTHESIS. Unless stated otherwise, all convex bodies considered here are assumed to be regular.

Examples. **1.** The simplest example is the solid unit disk D^n in $\mathbb{R}^n$, which is defined by the inequality

$$1 - \sum_i x_i^2 \geq 0 .$$

2. Clearly we want a subspace like the hypercube defined by $-1 \leq x_i \leq 1$ to be a convex body. This and many other examples will follow from a few simple observations which we shall now describe.

PROPOSITION. *Let K be a compact convex subset of $\mathbb{R}^n$ with nonempty interior such that K is defined by a finite set of inequalities $h_j(v) \geq 0$ where each h_j is a smooth real valued function such that $h_j(v) = 0$ implies $\nabla h_j(v) \neq \mathbf{0}$. Then h is a regular convex body.*

Proof. Let h be the minimum of the functions h_j. Then K is the set of points where $h(v) \geq 0$. If $h(v) > 0$ then $h_j(v) > 0$ for all j and in fact there is an open neighborhood V of v in $\mathbb{R}^n$ such that each h_j is positive on V. Therefore v is an interior point of K. If $h(v) = 0$, then $h_i(v) = 0$ for some i. By the Inverse/Implicit Function Theorem, we know that every open neighborhood of v contains points z such that $h_i(z) < 0$, and therefore v must be a frontier point of K.■

The preceding result holds for the hypercube, and hence the latter is a regular convex body as defined above. More generally, the result applies if each h_j is a first degree polynomial in the coordinates. This yields examples like the following:

3. The n-simplex $\Sigma_n \subset \mathbb{R}^n$ consisting of all points $(x_1, \cdots, x_n)$ such that each $x_i \geq 0$ and $\sum_i x_i \leq 1$.

4. The prism in $\mathbb{R}^{n+1}$ consisting of all points $(x_1, \cdots, x_n, x_{n+1})$ such that $(x_1, \cdots, x_n)$ lies in Σ_n and $0 \leq x_{n+1} \leq 1$.

The following theorem is intuitively what one would expect, but it plays an important role in many contexts:

THEOREM. *If K is a regular convex body in $\mathbb{R}^n$, then there is a homeomorphism from (D^n, S^{n-1}) to $(K, \partial K)$.*

Proof. First of all, we claim it suffices to consider regular convex bodies such that $\mathbf{0} \in \mathbb{R}^n$ lies in the interior. If K is a regular convex body which contains the point p and T is the isometry of $\mathbb{R}^n$ sending x to $x - p$, then $K' = T[K]$ is also a regular convex body, but it contains the zero vector in its interior, and if the conclusion of the theorem holds for $(K', \partial K')$ then it clearly also holds for $(K, \partial K)$.

Assuming now that $\mathbf{0}$ lies in the interior of K, we know that there is some $\varepsilon > 0$ such that the open ε-disk centered at $\mathbf{0}$ lies in the interior of K. Let $v \in S^{n-1}$ be given, and consider the intersection of K with the closed ray $L(v)$ consisting of all points of the form tv, where $t \geq 0$. Then $K \cap L(v)$ is a closed bounded convex set containing all points tv for $t \leq \varepsilon$, and therefore it follows that $K \cap L(v)$ must be a close interval consisting of all tv where $0 \leq t \leq b(v)$ for some $b(v) > 0$.

CLAIM: The point $b(v)\,v$ is the unique point in $\partial K \cap L(v)$.

To prove the claim, first note that the intersection $\partial K \cap L(v)$ is a closed bounded subset of $\mathbb{R}^n$, and since it is disjoint from an open neighborhood of $\mathbf{0}$ there will be a least positive number $a(v)$ such that $a(v)\,v$ lies in that intersection. The assertion in the claim is equivalent to saying that $a(v) = b(v)$; we shall prove that $a(v) < b(v)$ leads to a contradiction. If this condition holds, then choose $a'(v)$ such that $0 < a'(v) < a(v)$, and let $\eta > 0$ such that the open η-disk centered at $a'(v)$ is contained in the interior of K. Let E denote the disk consisting of all points w such that $w = w_0 + a(v)\,v$, where w_0 is perpendicular to v and $|w_0| < \frac{1}{2}\,\eta$. By convexity the set K contains all convex combinations of the form $t\,b(v)\,v \ + \ (1-t)w$, where w is as above and $0 \leq t \leq 1$. If M denotes this set, then M is a cone which contains the point $a(v)\,v$ in its interior; this can be proved in a variety of ways, and one argument is sketched at the end of this document. Since we assumed that $a(v)\,v$ was a frontier point of K, we have derived a contradiction, and therefore we must have $a(v) = b(v)$, proving the claim.

The next step is to prove that $b(v)$ is a continuous function of $v \in S^{n-1}$. Since $\mathbf{0}$ lies in the interior of K, it follows that $b(v) \geq \varepsilon$, where ε is as above. Furthermore, since K is bounded it follows that $b(v)$ is bounded from above by some constant. Thus we have a well-defined function b from S^{n-1} to some closed interval $[m, M]$ for suitable constants satisfying $M > m > 0$.

Under the standard radial homeomorphism from $\mathbb{R}^n - \{\mathbf{0}\}$ to $S^{n-1} \times (0, \infty)$, the boundary ∂K corresponds to the graph of b. Since ∂K is a closed subset of $\mathbb{R}^n$, the continuity of b will be an immediate consequence of the following result:

LEMMA. (Closed graph property) *Let $f : X \to Y$ be a map of compact Hausdorff spaces. Then f is continuous if and only if its graph is a closed subset of $X \times Y$.*

Proof. Suppose that X and Y are Hausdorff and f is continuous. Let $F : X \times Y \to Y$ and $G : X \times Y \to Y$ be the functions $F(x, y) = y$ and $G(x, y) = f(x)$. Then the graph of f (= the set of (x, y) such that $y = f(x)$) is the set of all points where $F = G$. Since this set of points is closed for maps into a Hausdorff space, it follows that the graph is closed in he product.

Now assume both spaces are also compact and that the graph Γ of f is a closed subset of the product. Then the map f factors into a composite of $\gamma(f) : X \to X \times Y$ — which is defined by $\gamma(f)(x) = (x, f(x))$ — and the projection $P : X \times Y \to Y$ onto the second coordinate. Let Γ denote the image of $\gamma(f)$, so that Γ is a compact subset of the product; it follows that $\gamma(f)$ defines a 1–1 correspondence γ' from X to Γ. If $Q : X \times Y \to X$ is projection onto the first coordinate and $g = Q|\Gamma$, then g is continuous and is an inverse to γ'. But since Γ and X are compact Hausdorff, the map g must be a homeomorphism, so that its inverse, which is γ' must be continuous. The latter implies that $\gamma(f)$ is continuous, which in turn implies that $f = P \circ \gamma(f)$ is also continuous.■

EXAMPLE. The preceding result does not extend to noncompact spaces. For example, let X be the nonnegative integers with the usual metric, and let Y be the set of all points on the real line of the form 0 or $1/n$ for some positive integer n. Then the map $f : X \to Y$ sending 0 to itself and $n > 0$ to $1/n$ is continuous because every map from a discrete space is continuous. Clearly the map is also 1–1 and onto, but its inverse is not continuous. But the graph of f^{-1} is the set of all (y, x) such that $y = f(x)$ (why?) and hence it is closed in $Y \times X$.

Completion of the proof of the theorem. Define the **radial projection mapping** $\rho : D^n \to K$ by $\rho(\mathbf{0}) = \mathbf{0}$ and for nonzero points of the form tv where $0 < t \leq 1$ and $v \neq 0$ define $\rho(tv) = t\, b(v)\, v$. It follows immediately that this map is 1–1 onto and continuous except possibly at $\mathbf{0}$. To see continuity at $\mathbf{0}$, let M_0 be the maximum value of the function b, and let $h > 0$. Then $\rho(x) \leq M_0|x|$ holds, and therefore we know that $|x| < h/M_0$ implies $|\rho(x)| < h$, proving continuity at $\mathbf{0}$. Since D^n is compact and K is Hausdorff, it follows that ρ must be a homeomorphism.■

Appendix

We shall prove the following result, which was one step in the proof of the main theorem:

PROPOSITION. *Let a be a nonzero vector in $\mathbb{R}^n$, let b be a vector in $\mathbb{R}^n$, let $\delta > 0$ be a positive real number, and let E be the $(n-1)$-dimensional disk consisting of all points $w = b + w_0$, where w_0 is perpendicular to a and $|w_0| \leq \delta$. Let H be the smallest convex set containing E and $a + b$. Then H contains all points of the form $b + t\,a$, where $0 < t < 1$, in its interior.*

In the application to the proof of the main theorem, we take $b = a'(v)\, v$ and $a = \big(\, b(v) - a'(v)\,\big)\, v$. The point $a(v)\, v$ then can be rewritten as

$$a(v)\, v \quad = \quad a'(v)\, v \ + \ \frac{a(v) - a'(v)}{b(v) - a'(v)} \big(\, b(v) - a'(v)\,\big)\, v$$

and since

$$0 \quad < \quad \frac{a(v) - a'(v)}{b(v) - a'(v)} \quad < \quad 1$$

this translates to the equation $a(v)\, v \ = \ b + t\,a$ were $0 < t < 1$ and hence the point $a(v)\, v$ lies in the interior, which was the objective.

Proof. The first step is to reduce this to a case where a and b have particularly simple forms. Specifically, if $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ is translation by $-a$, then T_1 transforms the entire picture into one for which $a = 0$, and since we have

$$T_1(s\,u \ + \ (1-s)v) \quad = \quad s\,T_1(u) \ + \ (1-s)\,T_1(v)$$

for all u and b (verify this!), it follows that $T_1[H]$ is the smallest convex set containing $T_1(a+b) = b$ and the disk $T_1[E]$; in effect, this reduces everything to proving the result when $b = 0$. Similarly, if T_2 is an orthogonal transformation sending b to a positive multiple of the last unit vector $\mathbf{e}_n = (0, ..., 1)$, we can reduce everything to the case where $a = k\,\mathbf{e}_n$ for some $k > 0$.

If a and b are given as above, then we claim that H is the solid cone consisting of all points $(x_1, \cdots, x_n)$ satisfying the inequalities $x_n \geq 0$ and

$$\frac{x_n}{k} \quad \leq \quad 1 \ - \ \frac{1}{\delta} \sqrt{\sum_{i=1}^{n-1} x_i^2} \ .$$

If this is true, then the points specified the corresonding strict inequality (with $>$ replacing $\geq$), which defines an open subset of the solid cone, and hence these point must lie in the interior of H as claimed.

Consider a typical 2-dimensional section H' of H obtained by intersecting it with a the 2-dimensional vector subspace P spanned by $\mathbf{e}_n$ and some unit vector $\mathbf{u}$ perpendicular to $\mathbf{e}_n$; there is a drawing in the file `convexbodies2.pdf`. Elementary considerations show that H' must contain the solid triangle in P with vertices $y\,\mathbf{e}_n$ and $\pm\,\delta\,\mathbf{u}$, which is the smallest convex set containing the given three vertices. The union of these plane sections is just the cone described by the preceding inequality, and therefore H must contain this solid cone. Finally, straightforward computation implies that this solid cone is convex, and therefore H must be the solid cone.■