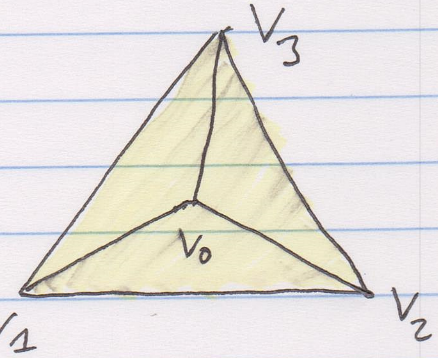
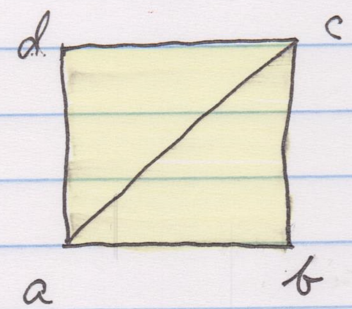
Examples

Star shaped simplicial complexes



Star shaped with respect to v_0

$(v_0 < v_1 < v_2 < v_3)$



$a < b < c < d$

Star shaped with respect to a.

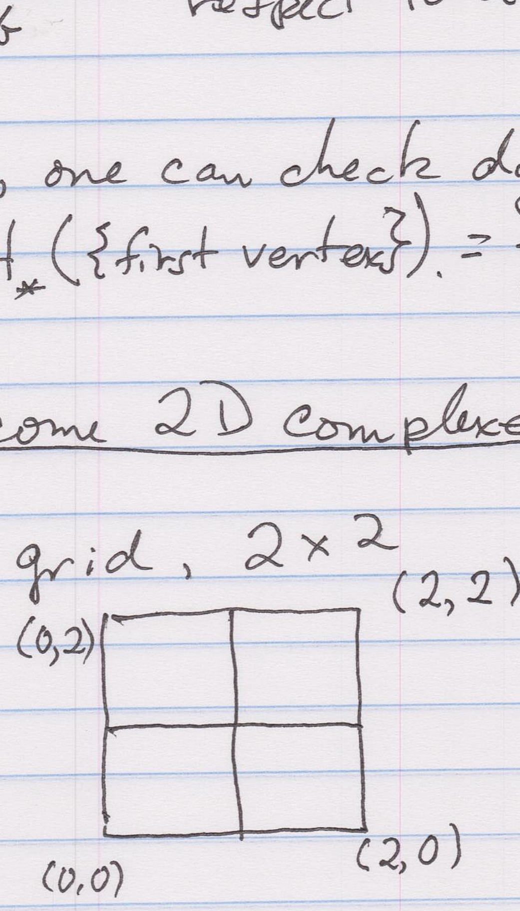
In each case, one can check directly that

$$H_*(K^\omega) = H_*(\{\text{first vertex}\}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * \neq 0 \end{cases}$$

Homology of some 2D complexes

Start with 1D grid, 2×2

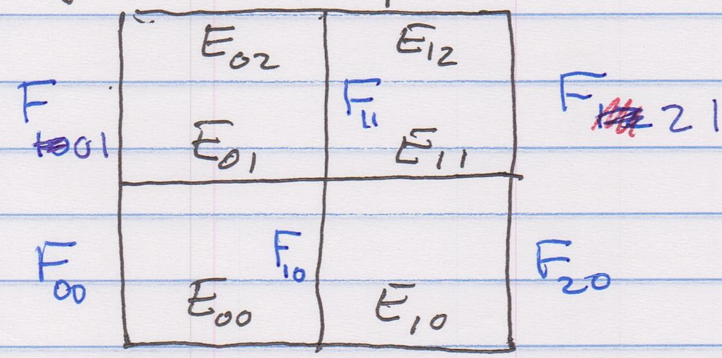
Order vertices lexicographically.



$H_0 = \mathbb{Z}$

$H_1 = \mathbb{Z}^4$.

Free generators for H_1 are given by cycles representing the four squares

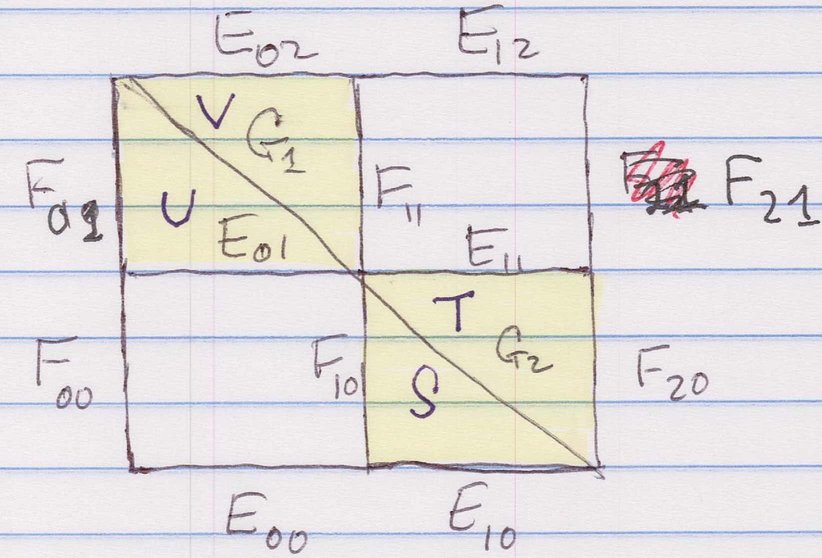


$$E_{00} + F_{10} - E_{01} - F_{00},\quad E_{10} + F_{20} - E_{11} - F_{\cancel{00}\,10}$$

$$E_{01} + F_{11} - E_{02} - F_{\cancel{10}\,01},\quad E_{11} + F_{\cancel{12}\,21} - E_{12} - F_{11}.$$

<u>Checkerboard complex</u>

S, T, U, V are the 2-simplices



Again, $H_0 = \mathbb{Z}$, but

$H_1 = \mathbb{Z}^2$ (2nd & 4th gens go to zero)

$H_2 = 0$ (d_2 is 1-1 — verify this!)

$H_2 = \operatorname{Ker} d_2$ since $C_3 = 0$

To see d_2 is 1-1, note that if we have a 2-chain

$aS + bT + cU + eV$, then

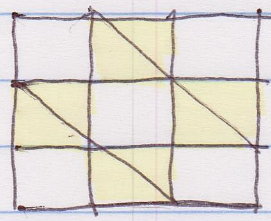
in $d(aS + bT + cU + eV)$ we have

(i) coeff of F_{10} is $\pm a$
(ii) coeff of F_{20} is $\pm b$
(iii) coeff of F_{11} is $\pm c$
(iv) coeff of F_{01} is $\pm e$

Hence $d(\text{2-chain}) = 0 \Longrightarrow$

$$a = b = c = e = 0.$$

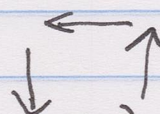
<u>Question</u> What happens if we do the same sort of thing in the 3×3 case? What about $n \times n$?



$\boxed{3 \times 3 \text{ case}}$

The first thing to do is make an educated guess about the groups H_0, H_1, H_2.

$H_0 = \mathbb{Z} \rightsquigarrow$ H_0 only depends upon the subcomplex of 0- and 1- simplices, and that subcomplex is connected, so $H_0 \cong \mathbb{Z}$ as in the case of graphs.

$H_1 \cong \mathbb{Z}^5 \rightsquigarrow$ In the 3×3 grid we have $H_1 \cong \mathbb{Z}^9$, and 4 of the 9 holes are filled in by squares. Each $\square$ yields a 2-chain s.t. $d_2(\text{chain}) =$  .

We should also expect $H_2 = 0$ for similar reasons to those in the 2×2 case.

Generalizing to $n \times n$ should now be simple, at least intuitively: For the grid one has $H_0 \cong \mathbb{Z}$, $H_1 \cong \mathbb{Z}^{n^2}$, and one is filling in half of the holes — more

n^2+1 precisely, $\left[\frac{\cancel{n^2}}{2}\right]$ of them, where $[x] =$

$\frac{n^2}{2}$ if n even, greatest integer $\leq x$. Hence H_1 of the

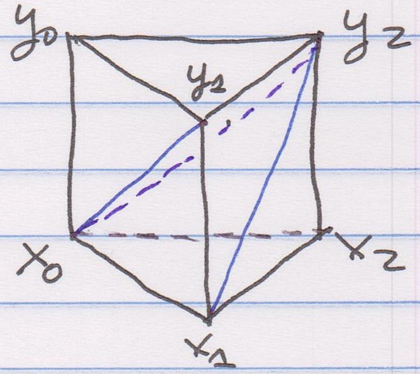
$\frac{n^2+1}{2}$ if n odd checkerboard complex should be

n^2+1, $\mathbb{Z}^{n^2 - \left[\frac{\cancel{n^2}}{2}\right]}$. Also, we expect

$H_2 = 0$.

One can check these guesses are true by brute force, but of course we would like to have better ways of doing so. Not surprisingly, we have to develop some machinery for this purpose.

<u>A nonplanar example</u> Look at the boundary of a triangular prism.



We have a simplicial decomposition with vertices as shown and 8 2-simplices

$x_0x_1x_2$, $x_0x_1y_1$, $x_0y_0y_1$, $x_0x_2y_2$, $y_0y_1y_2$, $x_1y_1y_2$, $x_1x_2y_2$, $x_0y_0y_2$.

$H_0 \cong \mathbb{Z}$ by connectedness.

<u>CLAIM</u> $H_1 = 0$, $H_2 \cong \mathbb{Z}$.

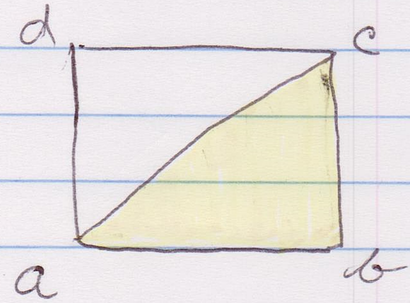
An explicit generator for H_2 is

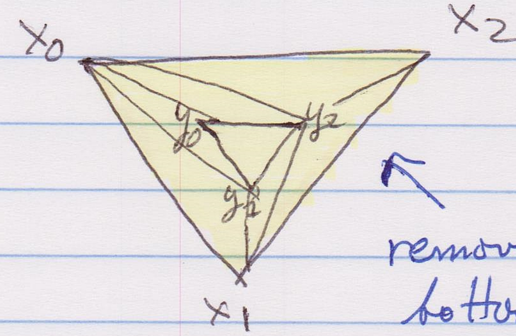
$$x_0x_1x_2 - x_0x_1y_1 + x_0y_0y_1 + x_0x_2y_2$$
$$- y_0y_1y_2 + x_1y_1y_2 - x_1x_2y_2 - x_0y_0y_2$$

(check this out!).

Again, we need a better way to do such things.

Related example: What happens if we remove a 2-simplex?





remove the bottom 2-simplex $x_0 x_1 x_2$ and flatten what's left.

Educated guesses

LHS $H_1 \cong \mathbb{Z}$, $H_2 = 0$ **RHS** $H_1 \cong H_2 \cong 0$.

More generally $K = L \cup n$-simplex, how to compare $H_*(K)$ and $H_*(L)$?

As usual, suppose n = dimension of K.