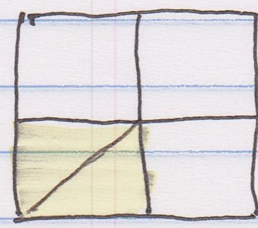
EXAMPLES REVISITED

Homology of checkerboard complexes

We start with an $n \times n$ grid graph as before and fill in squares one at a time.

First step



$K \cap L = \text{bdy } M.$

$L = \text{Grid}, \quad K = L \cup M \qquad M = \text{Square}.$

Use the Mayer-Vietoris sequence

$$\begin{array}{ccccccc} H_2(L) & & & & & & H_1(L) \\ \oplus & \longrightarrow & H_2(K) & \longrightarrow & H_1(\text{bdy } M) & \longrightarrow & \oplus \\ H_2(M) & & & & & & H_1(M) \end{array}$$

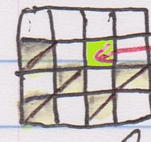
$$\begin{array}{ccccc} & & & & H_0(L) \\ \longrightarrow H_1(K) & \longrightarrow & H_0(\text{bdy } M) & \longrightarrow & \oplus \\ & & & & H_0(M) \end{array}$$

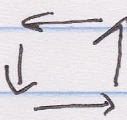
Insert known values

$$\begin{array}{ccccccccc} & & & & & & \mathbb{Z}^{n^2} & & \\ 0 & \longrightarrow & H_2(K) & \longrightarrow & \mathbb{Z} & \longrightarrow & \oplus & \longrightarrow & H_1(K) \\ & & & & & & 0 & & \end{array}$$

$$\begin{array}{ccc} & & \mathbb{Z} \\ \longrightarrow H_0 = \mathbb{Z} & \xrightarrow{(1,-1)} & \oplus \\ & & \mathbb{Z} \end{array}$$

We know that the generator of $H_1(M) \cong \mathbb{Z}$ goes to one of the free generators for $H_1(L)$, and by exactness $H_1(K) \longrightarrow H_0(\text{bdy}\, M)$ is zero. These imply that $H_2(K) = 0$ and $H_1(K) = \mathbb{Z}^{n^2}$.



<u>Inductive step</u> Suppose we have filled in k squares and know that the resulting complex L_k has $H_2 = 0$, $H_1 \cong \mathbb{Z}^{n^2-k}$ s.t. the latter has the expected free generators,  for each unfilled square. Consider $L_{k+1} = L_k \cup M_{k+1}$; as before $\text{bdy}\, M_{k+1} = L_k \cap M_{k+1}$. We then have the following M-V sequence:

$$\begin{matrix} H_2(L_k) \\ \oplus \\ H_2(M_{k+1}) \end{matrix} \longrightarrow H_2(L_{k+1}) \longrightarrow H_1(\text{bdy}) \longrightarrow \begin{matrix} H_1(L_k) \\ \oplus \\ H_1(M_{k+1}) \end{matrix} \longrightarrow H_1(L_{k+1}) \xrightarrow{\Delta}$$

$0 \qquad\qquad \mathbb{Z} \qquad\qquad \cong \mathbb{Z}^{n^2-k} \qquad\qquad$ zero map.

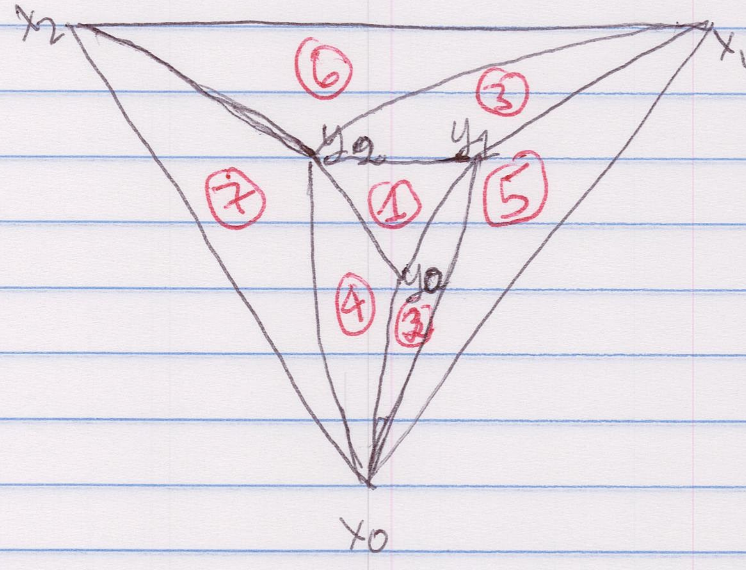
We get $\Delta = 0$ because the next map in the sequence is

$$\mathbb{Z} = H_0(M_{k+1}) \xrightarrow{(1,-1)} \begin{matrix} H_0(L_k) \\ \oplus \\ H_0(M_{k+1}) \end{matrix} = \begin{matrix} \mathbb{Z} \\ \oplus \\ \mathbb{Z} \end{matrix}.$$

(a 1-1 map)

Hence $H_2(L_{k+1}) = 0$, $H_1(L_{k+1}) = \mathbb{Z}^{n^2-(k+1)}$.

<u>Boundary of triangular prism</u>

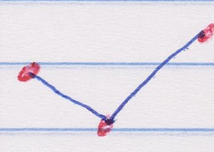


$L =$ subcomplex of all 2-simplices except bottom one: $x_0 x_1 x_2$

Let L_k = subcomplex given by the first k simplices $A_1 \cup \dots \cup A_k$, so

$$L_{k+1} = L_k \cup A_{k+1}$$

$L_k \cap A_{k+1}$ = edge or two joined edges



Show $H_*(L_k) = H_*(\{\text{vertex}\})$ inductively using M-V sequences. L_1 is a 2-simplex, so the statement is true when $k=1$.

$$H_1(L_{k+1}) \xrightarrow{\Delta} \underset{\mathbb{Z}}{H_0(\text{intersection})} \xrightarrow{(1,-1)} \begin{array}{c} H_0 \\ \oplus \\ H_0 \end{array} = \begin{array}{c} \mathbb{Z} \\ \oplus \\ \mathbb{Z} \end{array}$$

So $\Delta = 0$. Now the next term on the left is $\begin{array}{c} H_1(L_k) \\ \oplus \\ H_1(A_{k+1}) \end{array} = \begin{array}{c} 0 \\ \oplus \\ 0 \end{array}$, so $H_1(L_{k+1}) = 0$.

In higher dims we have $(q \geq 2)$

$$\underset{\underset{0}{\shortparallel}}{H_q(\text{int.})} \longrightarrow \underset{\underset{0 \oplus 0}{\shortparallel}}{\begin{array}{c} H_q(L_k) \\ \oplus \\ H_q(A_{k+1}) \end{array}} \longrightarrow H_q(L_{k+1}) \longrightarrow \underset{\underset{0}{\shortparallel}}{H_{q-1}(\text{int.})}$$

so $H_q(L_{k+1}) \cong H_q(\text{pt.})$.

At the final step, where K = the entire prism body, we have

$$K = L_7 \cup A_8 \qquad A_8 = x_0 x_1 x_2$$

$$L_7 \cup A_8 = \text{Bdy } A_8 \simeq T \text{ (triangle)}.$$

The M-V sequences are now a little different.

$H_1(L_7) = H_1(A_8) = 0$ still implies $H_1(K) = 0$, but if $q \geq 2$ then we have

$$\begin{matrix} H_q(L_7) \\ \oplus \\ H_q(A_8) \\ \shortparallel \\ 0 \oplus 0 \end{matrix} \longrightarrow H_q(K) \xrightarrow{\Delta} H_{q-1}(T) \longrightarrow \begin{matrix} H_{q-1}(L_7) \\ \oplus \\ H_{q-1}(A_8). \\ \shortparallel \\ 0 \oplus 0. \end{matrix}$$

Hence Δ is an isomorphism.

Since $H_m(T) = \begin{matrix} \mathbb{Z} & m = 0,1 \\ 0 & m \geq 2 \end{matrix}$, this yields

$$H_q(K) = \begin{cases} \mathbb{Z} & q = 0, 2 \\ 0 & \text{otherwise}. \end{cases}$$