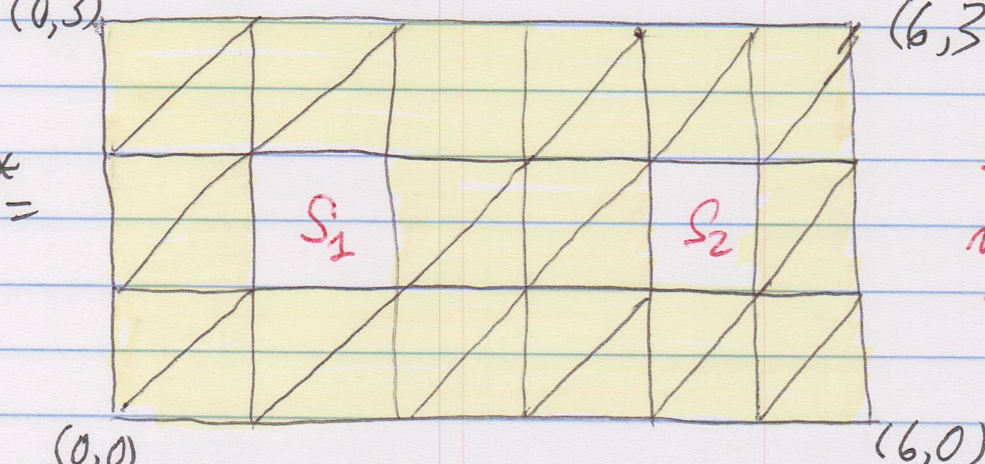
Yet Another Example

A rectangle with two square holes

$M^* =$



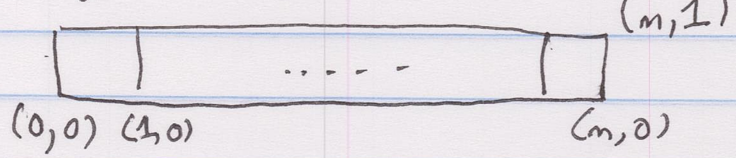
So S_1, S_2 not in M^*, except for bdries.

Start with the rectangular grid whose vertices are the pairs in $\{0, \dots, 6\} \times \{0, \dots, 3\}$. Fill in all the squares except for the two whose lower left corners are $(1,1)$ and $(4,1)$.

and add diagonal edges to get a simplicial decomp.

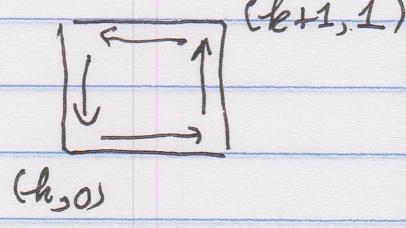
We shall start with a general version of some observations in the first document in this series.

$n \times 1$ grid K_n



order the vertices lexicographically

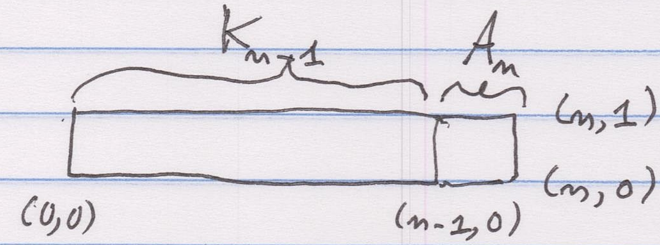
<u>Claim</u> $H_1(K_n) = \mathbb{Z}^n$, generators



$E(v;w)$ = edge with endpts $v < w$

$$E((k,0);(k+1,0)) + E((k+1,0);(k+1,1))$$
$$- E((k,1);(k+1,1)) - E((k,0);(k+1,0))$$

One way to verify this is by induction on n:



$\boxed{n = 1 \text{ easy to check}}$

$K_{n-1} \cap A_n$ = 1-simplex. As before

$H_0(K_n) = 0$ (connected)

M-V for $K_n = K_{n-1} \cup A_n \Longrightarrow$

$$H_q(K_n) \cong H_q(K_{n-1}) \oplus H_q(A_n) \quad (\text{ONLY } q = 1)$$

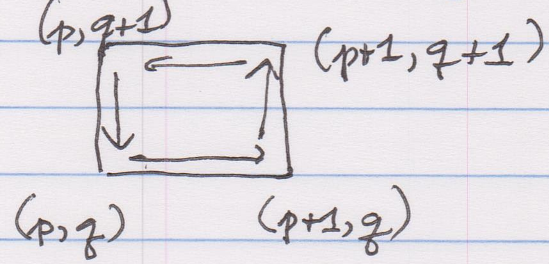
since $H_1(K_{n-1} \cap A_n) = 0$ and

$H_0(K_{n-1} \cap A_n) \longrightarrow H_0 \oplus H_0$ is 1-1.

Now consider $n \times m$ grids. <u>CLAIM</u>

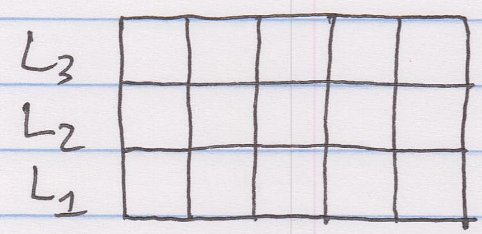
$H_1 = \mathbb{Z}^{mn}$, free generators

$\alpha_{p,q}$



Let $L_1, \dots, L_m$ be the $n \times 1$ subgrids.

$m = 3$



$K_r = L_1 \cup \dots \cup L_r$. Prove the claim by induction on r; $r = 1$ was done above. If the result is true for $r-1$, look at $K_r = K_{r-1} \cup L_r$. Notice that

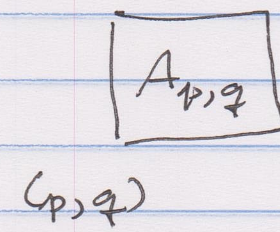
$K_{r-1} \cap L_r \cong$ 1-dim cpx $\underset{0}{\bullet}\!-\!\underset{1}{\bullet}\cdots\!-\!\underset{n}{\bullet}$ M_0

for which $H_*(M) \cong H_*(\{0\})$ by induction on n (or notice that M_0 is a tree!!).

Another M-V argument as before shows that $H_1(K_r) \cong H_1(K_{r-1}) \oplus H_1(L_r)$.

and this completes the inductive step.

<u>The next step.</u> Suppose we fill in the squares $A(p_i, q_i)$



where $(p_i, q_i) =$ LOWER LEFT VERTEX

Then $H_1(\text{grid} \cup \text{squares}) = H_1(\text{grid}) / \langle \alpha_{p_i, q_i} \rangle$. | $H_2(\text{grid} \cup \text{squares}) = 0$

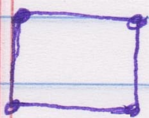
<u>Derivation</u> Proceed by induction on the number of squares to be filled in. No squares, no problem. Suppose we are filling in k squares and the result is known for filling in

$X_{k-1} \cap Y_k$ $(k-1)$ squares. Let $X_{k-1} =$ subcomplex

$= \text{Bdy}\, Y_k$. with first $(k-1)$ squares filled, $Y_k = k$th square.

Now look at the M-V sequence:

$$0 = H_1(X_{k-1} \cap Y_k) \longrightarrow \begin{matrix} H_2(X_{k-1}) \\ \oplus \\ H_2(Y_k) = 0 \end{matrix} \longrightarrow H_2(X_k)$$



$\leftarrow$ 1-dim complex

Bdy Y_k

Δ_2

$$H_1(X_{k-1} \cap Y_k) \longrightarrow \begin{matrix} H_1(X_{k-1}) = \text{free abel} \\ \oplus \\ H_1(Y_k) = 0 \end{matrix} \longrightarrow H_1(X_k)$$

$H_1(X_{k-1} \cap Y_k) = \mathbb{Z}$

$\Delta_1 = 0$

$$H_0(X_{k-1} \cap Y_k) \xrightarrow[\text{1-1}]{} \begin{matrix} H_0(X_{k-1}) & \mathbb{Z} \\ \oplus & \oplus \\ H_0(Y_k) & \mathbb{Z} \end{matrix}$$

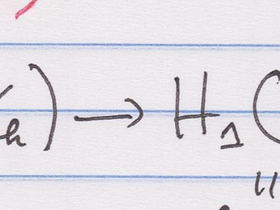
$H_0(X_{k-1} \cap Y_k) = \mathbb{Z}$

(as before)

We know that the image $H_1(X_{k-1} \cap Y_k) \longrightarrow H_1(X_{k-1})$

$H_1(X_{k-1} \cap Y_k) = \mathbb{Z}$, $H_1(X_{k-1}) = \text{free abel}$

is the cyclic subgroup generated by the cycle with representative



Bdy Y_k

so this implies that the map in homology is 1-1. The latter has two implications

① $\Delta_2 = 0$ by exactness, so that

$H_2(X_{k-1}) = 0$ (induction) implies

$H_2(X_k) = 0$ also by exactness.

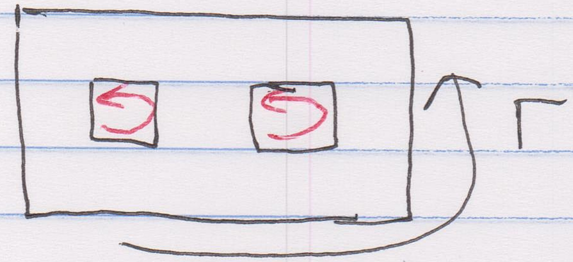
② $H_1(X_k) \cong H_1(X_{k-1}) / H_1(\text{Bdy } Y_k)$.

This completes the inductive step.

If we apply this to the complex on page 1, we see that $H_2 = 0$ and $H_1 \cong \mathbb{Z} \oplus \mathbb{Z}$ with free generators $\alpha_{1,1}$ and $\alpha_{4,1}$.

<u>Application</u> Let Γ be the cycle in this complex given by going around the boundary in the counterclockwise sense. Then the homology class of Γ is $[\alpha_{1,1}] + [\alpha_{4,1}]$.

M^*



<u>One derivation</u> Check that in the grid we have $\Gamma = \sum \alpha_{i,j}$, so $[\Gamma] = \sum [\alpha_{i,j}]$ in $H_1(\text{grid})$. By the preceding discussion, the image of $[\alpha_{i,j}]$ in our complex M^* is zero unless (i,j) is $(1,1)$ or $(4,1)$. Hence the RHS's image in $H_1(M^*)$ is just $[\alpha_{1,1}] + [\alpha_{4,1}]$.

So the outer boundary is homologous to the two pieces of the inner boundary, provided everything has a counterclockwise sense.