Solutions to Homework 5.

(1) Prove the *Five Lemma:*

Consider a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \downarrow{\scriptstyle t_4} & & \downarrow{\scriptstyle t_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

and prove:

(a) [4pts] If t_2 and t_4 are surjective and t_5 is injective, then t_3 is surjective.
(b) [4pts] If t_2 and t_4 are injective and t_1 is surjective, then t_3 is injective.
(c) [2pts] If t_1, t_2, t_4 and t_5 are isomorphisms, then t_3 is an isomorphism.

Proof. (a) Let $b_3 \in B_3$.

$\Rightarrow \exists a_4 \in A_4$ with $t_4(a_4) = g_3(b_3)$ (t_4 surjective)
$\Rightarrow g_4g_3(b_3) = 0 = g_4t_4(a_4) = t_5f_4(a_4)$
$\Rightarrow f_4(a_4) = 0$ (t_5 injective)
$\Rightarrow \exists a_3 \in A_3$ with $f_3(a_3) = a_4$
$\Rightarrow g_3(b_3 - t_3(a_3)) = g_3(b_3) - g_3t_3(a_3) = t_4(a_4) - t_4f_3(a_3) = t_4(a_4) - t_4(a_4) = 0$
$\Rightarrow \exists b_2 \in B_2$ with $g_2(b_2) = b_3 - t_3(a_3)$
$\Rightarrow \exists a_2 \in A_2$ with $t_2(a_2) = b_2$ (t_2 surjective)

Then

$$\begin{aligned} t_3(f_2(a_2) + a_3) &= t_3f_2(a_2) + t_3(a_3) \\ &= g_2t_2(a_2) + t_3(a_3) \\ &= g_2(b_2) + t_3(a_3) \\ &= b_3 - t_3(a_3) + t_3(a_3) \\ &= b_3 \end{aligned}$$

(b) Let $a_3 \in A_3$ with $t_3(a_3) = 0$.

$$\begin{aligned} &\Rightarrow t_4f_3(a_3) = g_3t_3(a_3) = 0 \\ &\Rightarrow f_3(a_3) = 0 \ \ (t_4 \text{ injective}) \\ &\Rightarrow \exists a_2 \in A_2 \text{ with } f_2(a_2) = a_3 \\ &\Rightarrow g_2t_2(a_2) = t_3f_2(a_2) = t_3(a_3) = 0 \\ &\Rightarrow \exists b_1 \in B_1 \text{ with } g_1(b_1) = t_2(a_2) \\ &\Rightarrow \exists a_1 \in A_1 \text{ with } t_1(a_1) = b_1 \ \ (t_1 \text{ surjective}) \\ &\Rightarrow g_1t_1(a_1) = g_1(b_1) = t_2f_1(a_1) = t_2(a_2) \\ &\Rightarrow f_1(a_1) = a_2 \ \ (t_2 \text{ injective}) \\ &\Rightarrow f_2(a_2) = f_2f_1(a_1) = 0 = a_3 \end{aligned}$$