Exact sequences

$$M \xrightarrow{f} N \xrightarrow{g} P$$

exact sequence $\iff$ Kernel g = Image f.

Generalize to longer sequences

$$A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \cdots \to A_m.$$

(each adjacent pair exact).

Prop. A chain complex is an exact sequence $\iff H_* = 0$.

Short exact sequence:

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

1-1 ONTO

$$C \cong B/A.$$

Similarly, can discuss short exact sequences of chain complexes.

<u>Example</u> $(Q,L) \subseteq (P,K)$
subcomplex

$$0 \to C_*(Q,L) \to C_*(P,K) \to C_*(K,L) \to 0$$

<u>Long Exact Sequence Theorem</u>

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \to 0$$

short exact sequence of chain complexes

$\Downarrow$

long exact homology sequence

$$\cdots \to H_{q+1}(C) \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(B) \xrightarrow{j_*} H_q(C) \xrightarrow{\partial} H_{q-1}(A) \cdots$$

Special cases

$\xrightarrow{0} A \xrightarrow{f} B$ exact $\iff$ f is 1-1

$A \xrightarrow{f} B \xrightarrow{0}$ exact $\iff$ f is onto

$\xrightarrow{0} A \xrightarrow{f} B \xrightarrow{0}$ exact $\iff$ f iso.

Multivariable Calculus Example

U = rectangular open set in $\mathbb{R}^2$

$a_1 \le x \le b_1$

$a_2 \le \cancel{x}\, y \le b_2$

(a_2, b_2)

$\boxed{U}$

(a_1, b_1)

$$0 \longrightarrow \mathbb{R} \xrightarrow[\text{fcns}]{\text{const}} C^\infty(U) \xrightarrow{\text{gradient}} \mathrm{Vec}(U) \xrightarrow{\text{scalar curl}} C^\infty(U)$$

smooth fcns on U

smooth vector fields on U

smooth fcns.

scalar curl: $F = (M, N) \longrightarrow \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$

$\nabla f = 0 \iff$ f constant

scalar curl $F = 0$ $\iff$ $F = \nabla g$ for some g.

<u>Quick application</u> Compute $H_*(\partial\Delta_n)$, say $n \geq 2$

<u>General comment</u> Previous considerations show $H_0(K) = 0$ if K is connected (i.e., P is connected) (Also see the exercises).

<u>Derivation</u> Look at the long exact sequence

$q \geq 2$

$$H_q(\Delta_n) \longrightarrow H_q(\Delta_n, \partial\Delta_n) \xrightarrow{\partial} H_{q-1}(\partial\Delta_n)$$
$$\downarrow$$
$$H_{q-1}(\Delta_n)$$

0   so ∂ is 1-1 and onto   0

$q = 1$ $$H_1(\Delta_n) \longrightarrow H_1(\Delta_n, \partial\Delta_n) \xrightarrow{\partial} H_0(\partial\Delta_n)$$
$$\cong \downarrow \text{ see above}$$
$$H_0(\Delta_n)$$

0   so $\partial = 0$, and

Thus $m \geq 1 \Rightarrow$

$$H_m(\partial\Delta_n) \cong H_{m+1}(\Delta_n, \partial\Delta_n)$$

which is

$\mathbb{Z}$ if $m+1=n$

0 otherwise.

Hence $\underset{n \geq 2}{H_q(\partial\Delta_n)} \cong \begin{cases} \mathbb{Z} & q=0, n-1 \\ 0 & \text{otherwise.} \end{cases}$

Proof of long exact sequence theorem — technique known as diagram or element chasing.

Need to define $\partial : H_q(C) \to H_{q-1}(A)$.

$$\begin{tikzcd}
 & & B_{q+1} \arrow[r] \arrow[d] & C_{q+1} \arrow[d] & \\
0 \arrow[r] & A_q \arrow[r, "i"] \arrow[d] & B_q \arrow[r, "j"] \arrow[d] & C_q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{q-1} \arrow[r] & B_{q-1} \arrow[r] & C_{q-1} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & y \arrow[r] \arrow[d] & x \arrow[d] \\
z \arrow[r] & dy \in A_{q-1} & 0
\end{tikzcd}$$

$[x] \in H_q(C)$ should go to

$[z] \in H_{q-1}(A)$. Show $dz = 0$

Is it a well-defined homomorphism?

Well-defined $x - x' = d\gamma \in C_{q+1}$

γ comes from $\beta \in B_{q+1}$. So $x - x' =$

$jd\beta = dj\beta$.

continue.

Fix x vary y Y.

$y - Y \to 0$

Choose x x'

Choose ~~y~~ Y ~~y'~~ Y'

Is this valid?

Well-defined? Check that the class in $H_{q-1}(A)$ does not depend upon the choices of x and y.

Homomorphism? Is ∂ a homomorphism?

First part ① For fixed choice x, the result does not depend upon the choice of y.
② The result does not depend upon the choice of x.

Second part "Coasting downhill."

Show ∂w does not depend upon the choice of y for a fixed choice of x.

Suppose $y, Y \longrightarrow x$; then $dY = i(Z)$ ~~for~~ some Z. But $y - Y \longrightarrow 0$ in C_*, so $y - Y = i(\alpha)$ some $\alpha \in A_q$. Hence $dy - dY = di(\alpha)$, or $Y \cancel{+} i(\alpha) = y$. Also $dY = i(Z) = di(\alpha) + dy = id\alpha + iz = i(z + d\alpha)$. Hence $Z = z + d\alpha$ and $[z] = [Z]$.

Show ∂w does not depend on the choice of x.

Say $x - x' = dw$, and choose $v \in A_{q+1}$ so $j(v) = w$. Then $d\cancel{y} - d\cancel{y}' = i(z) - i(z')$, y & y' s.t. $j(y) = x$, $j(y') = x'$. Compare $y - y'$ and dv. Both map to $x - x'$ under j [$jdv = djv = dw = x - x'$].

Hence $y - y' - dv = i(u)$, some $u \in A_{q+1}$. Compare dy and $dy' = dy - ddv - diu = dy - idu$.
So we have $i(z') = dy' = dy - idu = i(z - du)$, which means $z' = z - du$ and therefore $[z'] = [z]$.

<u>Homomorphism properties</u>. Given x, x', y, y', z, z' where $u = [x]$, $u' = [x']$. Check directly that $y + y' \longrightarrow x + x'$ and $z + z' \longrightarrow d(y + y')$ [why does this show additivity?]

If $r \in R$ [the underlying ring], then $ry \longrightarrow rx$ and $dry = r \cdot i(z)$, so $\partial(ru) = \partial[rx] = [rz] = r[z]$.