

Exact sequences

$$M \xrightarrow{f} N \xrightarrow{g} P$$

exact sequence $\Leftrightarrow \text{Kernel } g = \text{Image } f.$

Generalize to longer sequences

$$A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \cdots \rightarrow A_m.$$

(each adjacent pair exact).

Prop. A chain complex is an exact sequence $\Leftrightarrow H_* = 0.$

Short exact sequence:

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

$\begin{array}{ccccccc} & & & \uparrow & & \uparrow & \\ & & & \text{1-1} & & \text{ONTO} & \\ & & & & & & \end{array}$

$$C \cong B/A.$$

Similarly, can discuss short exact sequences of chain complexes.

Example $(Q, L) \subseteq (P, K)$
subcomplex

$$0 \rightarrow C_*(Q, L) \rightarrow C_*(P, K) \rightarrow C_*(K, L) \rightarrow 0$$

Long Exact Sequence Theorem

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \rightarrow 0$$

short exact sequence of chain complexes

\Downarrow
long exact homology sequence

$$\dots \rightarrow H_{q+1}(C) \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(B)$$

$$H_q(C) \xrightarrow{\partial} H_{q-1}(A) \dots$$

Special cases

$$0 \rightarrow A \xrightarrow{f} B \text{ exact} \Leftrightarrow f \text{ is 1-1}$$

$$A \xrightarrow{f} B \rightarrow 0 \text{ exact} \Leftrightarrow f \text{ is onto}$$

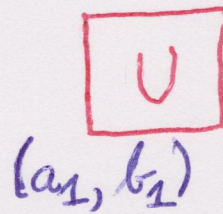
$$0 \rightarrow A \xrightarrow{f} B \rightarrow 0 \text{ exact} \Leftrightarrow f \text{ is iso.}$$

Multivariable Calculus Example

$U =$ rectangular open set in \mathbb{R}^2

$$a_1 \leq x \leq b_1$$

$$a_2 \leq y \leq b_2$$



$$0 \rightarrow \mathbb{R} \xrightarrow[\text{func}]{\text{const}} C^\infty(U) \rightarrow \text{Vec}(U) \rightarrow C^\infty(U)$$

Smooth func on U
Smooth vector fields on U
Smooth func.

$$\nabla f = 0 \Leftrightarrow$$

f constant

scalar curl $F = 0$

$$\Leftrightarrow F = \nabla g \text{ for some } g.$$

gradient

scalar curl

$$F = (M, N) \rightarrow$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Quick application Compute

$H_x(\partial\Delta_m)$, say $m \geq 2$

General comment Previous

considerations show $H_0(K) = 0$ if K is connected (i.e., P is connected) (Also see the exercises).

Derivation Look at the long exact sequence

$$\begin{array}{ccccc}
 & & q \geq 2 & & \\
 H_q(\Delta_m) & \rightarrow & H_q(\Delta_m, \partial\Delta_m) & \xrightarrow{\partial} & H_{q-1}(\Delta_m) \\
 \circ & & & & \downarrow \\
 & & \swarrow \text{so } \partial \text{ is 1-1} & & \circ H_{q-1}(\Delta_m) \\
 & & \text{and onto} & & \\
 q=1 & & & & \\
 H_1(\Delta_m) & \rightarrow & H_1(\Delta_m, \partial\Delta_m) & \xrightarrow{\partial} & H_0(\partial\Delta_m) \\
 \circ & & \text{so } \partial = 0, \text{ and} & & \cong \downarrow \text{see above} \\
 & & \text{also} & & H_0(\Delta_m)
 \end{array}$$

Thus $m \geq 1 \Rightarrow$

$$H_m(\partial\Delta_m) \cong H_{m+1}(\Delta_m, \partial\Delta_m)$$

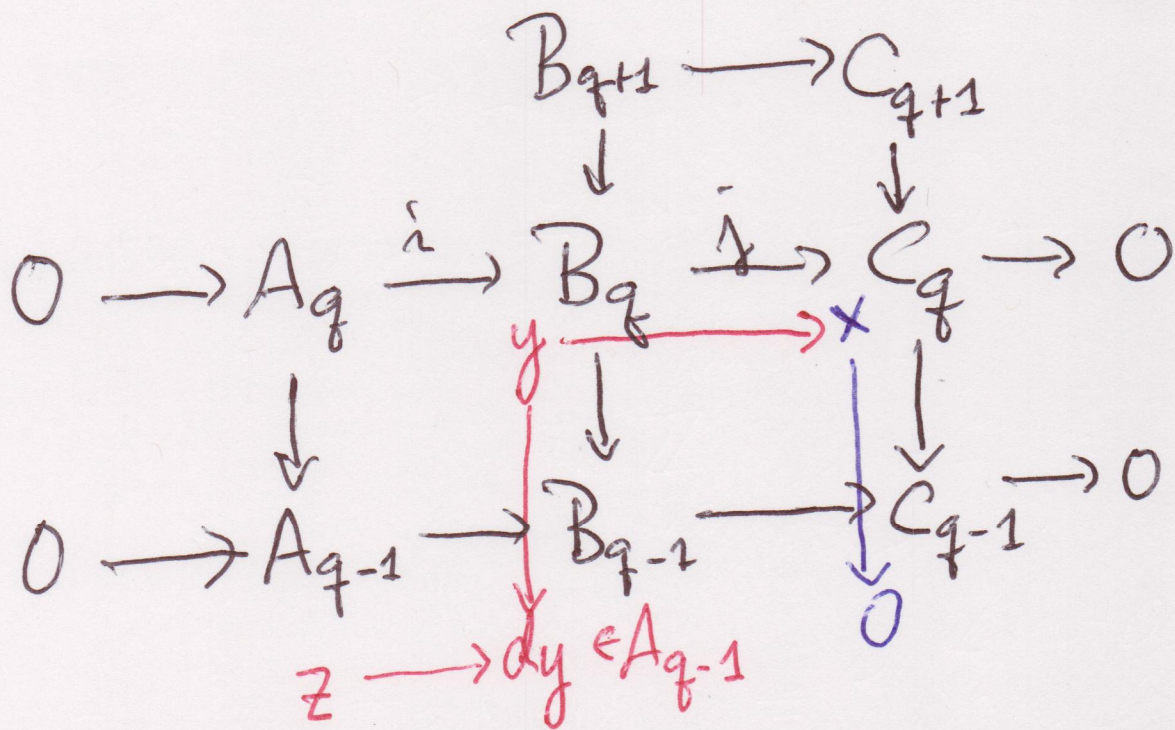
which is

$$\begin{cases} \mathbb{Z} & \text{if } m+1 = n \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Hence } H_q(\partial\Delta_m) \cong \begin{cases} \mathbb{Z} & q=0, m-1 \\ 0 & \text{otherwise.} \end{cases} \\ n \geq 2$$

Proof of long exact sequence
theorem — technique known
as diagram or element chasing.

Need to define $\partial: H_q(C) \rightarrow H_{q-1}(A)$.



$[x] \in H_q(C)$ should go to
 $[z] \in H_{q-1}(A)$. Should $z=0$

Is it a well-defined homomorphism?

Well-defined $x - x' = dd \in C_{q+1}$

γ comes from $\beta \in B_{q+1}$. So $x - x' = j d \beta = d j \beta$.

continue.

Choose x x'
~~Choose y y'~~

Fix x vary y
 $y - y' \rightarrow 0$

Is this valid?

Well-defined?

Check that the class in $H_{q-1}(A)$ does not depend upon the choices of x and y .

Homomorphism? Is ∂ a homomorphism?

First part ① For fixed choice x , the result does not depend upon the choice of y .

② The result does not depend upon the choice of x .

Second part "Coasting downhill."

Show \mathcal{I}_w does not depend upon the choice of y for a fixed choice of x .

Suppose $y, Y \rightarrow x$; then $dY = i(Z)$ for some Z . But $y - Y \rightarrow 0$ in C_x , so $y - Y = i(\alpha)$ some $\alpha \in A_q$. Hence $dy - dY = di(\alpha)$, or $Y \neq i(\alpha) = y$. Also $dY = i(Z) = di(\alpha) + dy = i d\alpha + i z = i(z + d\alpha)$. Hence $Z = z + d\alpha$ and $[z] = [Z]$.

Show \mathcal{I}_w does not depend on the choice of x .

Say $x - x' = dw$, and choose $v \in A_{q+1}$ so $j(v) = w$. Then $dy - dy' = i(z) - i(z')$, $y \neq y'$ s.t. $j(y) = x, j(y') = x'$. Compare $y - y'$ and dv . Both map to $x - x'$ under j [$j dv = djv = dw = x - x'$].

Hence $y - y' - dv = i(u)$, some $u \in A_{q+1}$. Compare dy and $dy' = dy - ddv - diu = dy - idu$. So we have $i(z') = dy' = dy - idu = i(z - du)$, which means $z' = z - du$ and therefore $[z'] = [z]$.

Homomorphism properties. Given x, x', y, y', z, z' where $u = [x], u' = [x']$. Check directly that $y + y' \rightarrow x + x'$ and $z + z' \rightarrow d(y + y')$ [why does this show additivity?]

If $r \in R$ [the underlying ring], then $ry \rightarrow rx$ and $dry = r \cdot i(z)$, so $\partial(ru) = \partial[rx] = [rz] = r[z]$.