**A FEW ADDITIONAL EXERCISES**

Here are some miscellaneous exercises that were not included in the course materials but might be useful to study in connection with the qualifying exam.

**1.** Prove that $A = ([-1,1] \times \{0\}) \cup (\{-1,1\} \times [0,1])$ is a strong deformation retract of $X = [-1,1] \times [0,1]$. [*Hint:* Let $p$ be the point $(0,2)$. For each $v = (x,y) \in X$ show that there is a unique point $a \in A$ such that $a = p + t(v - p)$ for some $t > 0$. Show that $t$ and hence $a$ are continuous functions of $v$; the formula for $t(v)$ is given by two different expressions depending on whether $2|x| + y \leq 2$ or $2|x| + y \geq 2$.]

**2.** Suppose that $G$ is a locally arcwise connected Hausdorff topological group and that the identity has a simply connected neighborhood.

($i$) Prove that the universal covering space $\widetilde{G}$ of $G$ is a topological group such that the projection $p : \widetilde{G} \to G$ is a continuous open homomorphism.

($ii$) Let $H$ be a connected topological group, and let $A$ be a discrete normal subgroup of $H$. Prove that $A$ is contained in the center of $H$.

($iii$) Show that the kernel of the map $p$ in ($i$) is a central subgroup if $G$ is connected.

**3.** Let $X$ be a connected, locally arcwise connected, semilocally simply connected, second countable (Hausdorff) space, and let $x \in X$. Prove that $\pi_1(X,x)$ is countable, and prove that every connected covering space of $X$ is second countable.

**4.** ($i$) Let $p : E \to X$ be a covering space projection, where $X$ is connected, locally arcwise connected, semilocally simply connected, and separable metric. Prove that $E$ is metrizable.

($ii$) Let $p : E \to X$ be a covering space projection, where $X$ is metrizable. Prove that $E$ is metrizable if and only if it is paracompact.

**5.**$^\star$ Suppose that $G$ is a Hausdorff topological group such that some neighborhood of the identity is metrizable. Prove that $G$ is metrizable if and only if $G$ is paracompact.