ALEXANDER DUALITY AND BORSUK'S SEPARATION CRITERION

This file expands upon some of the concepts and results discussed in `ahlfors.pdf`, and the goal is to show how one can derive the following result of K. Borsuk from the Alexander Duality Theorem and some related facts:

THEOREM. (Borsuk's Separation Criterion) *Let $A \subset S^2$ be a compact subset, and let x and y be two points in $W = S^2 - A$. Then the inclusion map from A into $S^2 - \{u, v\}$ is homotopic to a constant map if and only if x and y lie in the same component of U.*

A proof of this result by "elementary" methods is given by Lemmas 61.2 and 62.2 on pp. 377–378 and 382–383 of Munkres. One advantage of the proof by duality is that it can be modified to yield the analogous conclusion for subsets of S^n (*e.g.*, see the final chapter of the book by Eilenberg and Steenrod.

Further consequences of Alexander duality

In `ahlfors.pdf` we used one special case of the Alexander Duality Theorem for subsets of $\mathbf{R}^2$, and in this file we shall need a case in which the dimensions of the singular homology and Čech cohomology groups are switched.

Specifically, if $K \subset S^2$ is a compact (equivalently, closed) subset, then Alexander duality yields an isomorphism from the Čech cohomology group $H^1_{\check{C}}(K)$ to the reduced singular homology group $\widetilde{H_0}(S^2 - K)$, and these maps are natural with respect to closed subset inclusions $L \subset K$. By definition, the unreduced homology group is isomorphic to the free abelian group on the set of arc components, and the reduced homology group is the subset of all linear combinations in these arc components $\sum\, n_C\, C$ (where C runs through all the arc components) such that the coefficients satisfy $\sum\, n_C = 0$. The duality isomorphism is natural with respect to inclusions of subsets $L \subset K$; specifically, the inclusions of L in K and of $S^2 - K$ in $S^2 - L$ define homomorphisms such that the following diagram is commutative.

$$\begin{array}{ccc} \widetilde{H_0}(S^2 - K) & \longrightarrow & \widetilde{H_0}(S^2 - L) \\ \uparrow D_K & & \uparrow D_L \\ H^1_{\check{C}}(K) & \longrightarrow & H^1_{\check{C}}(L) \end{array}$$

By construction the unreduced 0-dimensional singular homology group $H_0(Y)$ of a space Y is a free abelian group on the set of arc components in Y. Let $p, q \in Y$, and let $\mathbf{g}(p)$ and $\mathbf{g}(q)$ denote the arc components of p and q. Given a continuous map $f : Y \to Z$, it follows that $f(p)$ and $f(q)$ lie in the same arc component of Z if and only if $f_*(\,\mathbf{g}(p)\,) = f_*(\,\mathbf{g}(q)\,)$. Furthermore, the class $\mathbf{g}(p) - \mathbf{g}(q)$ lies in the reduced homology group $\widetilde{H_0}(Y)$, and the preceding sentence implies that $f(p)$ and $f(q)$ lie in the same arc component if and only if $\mathbf{g}(p) - \mathbf{g}(q)$ lies in the kernel of the reduced homology map defined by f_*.

We now apply this to our original situation in which $L = A = S^2 - U$ and we are given points $x, y \in U$. It follows that we can find disjoint open neighborhoods W_1 and W_2 of x and y respectively such that $W_1 \cup W_2 \subset U$ and $B = K = S^2 - (W_1 \cup W_2)$ is homeomorphic to $S^1 \times [0, 1]$. We then have $L \subset K$, and if we take f to be the inclusion of $W_1 \cup W_2 = S^2 - K$ in $U = S^2 - L = S^2 - A$, it follows from the discussion in the preceding paragraph that x and y lie in the same component of U if and only if $\mathbf{g}(x) - \mathbf{g}(y)$ maps to zero in $\widetilde{H_0}(U)$. Since $W_1 \cup W_2$ has two arc components, it follows that $\widetilde{H_0}(W_1 \cup W_2)$ is infinite cyclic and is generated by $\mathbf{g}(x) - \mathbf{g}(y)$.

By Alexander duality and the basic properties of Čech cohomology (as developed in Eilenberg and Steenrod), we have the following commutative diagram

$$\begin{array}{ccccc} \mathbf{Z} & \cong & \widetilde{H_0}(W_1 \cup W_2) & \longrightarrow & \widetilde{H_0}(U = S^2 - A) \\ & & \uparrow D_B & & \uparrow D_A \\ \mathbf{Z} & \cong & H^1_{\check{C}}(B \cong S^1 \times [0,1]) & \longrightarrow & H^1_{\check{C}}(A) \end{array}$$

and hence it follows that $\mathbf{g}(x) - \mathbf{g}(y)$ maps to zero in $\widetilde{H_0}(U)$ if and only if the map $f^* : H^1_{\check{C}}(B) \to H^1_{\check{C}}(A)$ determined by the continuous mapping

$$f : A \xrightarrow{\text{inclusion}} B \;\cong\; S^1 \times [0,1] \xrightarrow{\text{projection}} S^1$$

sends a generator of $H^1_{\check{C}}(B)$ to the zero element of $H^1_{\check{C}}(A)$.

Bruschlinsky's Theorem

The final step in the proof of Borsuk's Separation Criterion is given by the following result:

BRUSCHLINSKY'S THEOREM. *Let X be a compact subset of $\mathbf{R}^n$ for some n, and let $f : X \to S^1$ be a continuous map. Then f is homotopic to a constant map if and only if the map f^* in 1-dimensional Čech cohomology is trivial.*

If we combine this we the previous discussion, we obtain Borsuk's Separation Criterion as a corollary.■

The underlying idea is to choose a generator $\omega \in H^1_{\check{C}}(X)$, to define a natural transformation from $[X, S^1]$ to $H^1_{\check{C}}(X)$ by sending a homotopy class $[f]$ to $f^*(\omega)$, and to prove that this natural transformation is an isomorphism. One reference for a formal statement and proof of Bruschlinsky's Theorem is the following standard text:

S.-T. Hu. *Homotopy Theory*, Pure and Applied Mathematics Series, Volume VIII. Academic Press, New York, 1959.

There are actually several approaches to proving Bruschlinsky's Theorem. For example, one can use the results of Hatcher's book to prove the result for subspaces of $\mathbf{R}^n$ that are finite cell complexes and to retrieve the general case by (i) describing every compact subset of $\mathbf{R}^n$ as a suitable (finite or countably infinite) intersection of finite cell complexes, (ii) using some basic naturality and "continuity" properties of the functors $[X, S^1]$ and $H^1_{\check{C}}(X)$.■