COMPUTING FUNDAMENTAL GROUPS OF SPECIFIC GRAPHS

We shall give two examples.

The Königsberg bridge graph. This graph reflects the locations of various bridges in the Baltic city of Königsberg (now Kaliningrad, Russia) during the 18^{th} century. Background information about this topic and the surrounding mathematics can be found at the following online sites (however, in this course we are not considering the original problem associated to these bridges):

`http://contracosta.edu/math/konig.htm`

`http://mathforum.org/isaac/problems/bridges1.html`

`http://mathforum.org/isaac/problems/bridges2.html`

`http://mathworld.wolfram.com/KoenigsbergBridgeProblem.html`

In this model, the bridges correspond to edges and the vertices to the land masses connected by various bridges. There are four basic vertices **a**, **b**, **c**, **d**, with one bridge from **a** to each of **b**, **c**, **d**, and two bridges joining each of **b** and **d** to **c**. This is not a linear graph in the sense of Munkres, but we can make it into a linear graph by taking the derived structure as in the commentaries, and by Additional Exercise 84.1 we know that the Euler characteristic is computable using the edges and vertices of the original decomposition. Thus the Euler characteristic of the underlying graph is given by $\chi = 4 - 7 = -3$, so that the fundamental group is a free group on $4 = 1 - \chi$ generators.

The edges of a cube. For the sake of definiteness, we shall consider the edges of the cube whose vertices are $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $\varepsilon_i = \pm 1$. There are 8 vertices, there are 4 horizontal edges in each of the planes $z = \pm 1$, and there are 4 additional vertical edges joining the vertices $(\varepsilon_1, \varepsilon_2, -1)$ and $(\varepsilon_1, \varepsilon_2, -1)$. It follows immediately that the Euler characteristic is $\chi = 8 - 12 = -4$ and that the fundamental group of the graph is a free group on $5 = 1 - \chi$ generators.

The Contra Costa site depicts several other examples of graphs. Looking at these examples and computing their fundamental groups is very strongly recommended.