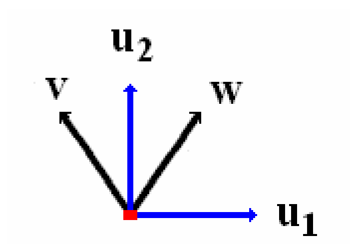
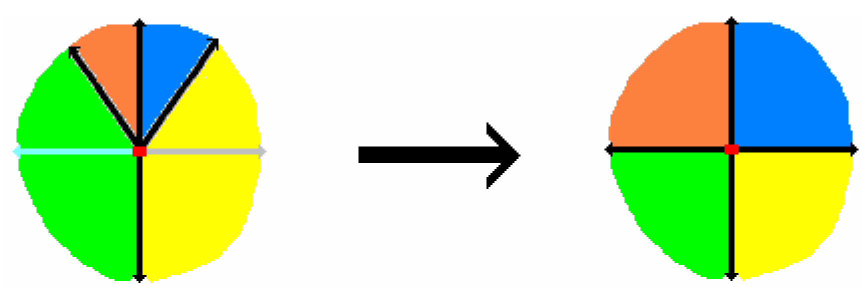
A Flattening Construction

In the discussion below, all vectors are assumed to lie in the coordinate plane.

We are given two unit vectors $\mathbf{v}$ and $\mathbf{w}$ that are linearly independent, and $\mathbf{u_2}$ is a unit vector which points in the same direction as $\mathbf{v} + \mathbf{w}$. The drawing below suggests that we can find a vector $\mathbf{u_1}$ in the span of $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{u_1}$ is perpendicular to $\mathbf{u_2}$ and the inner product of $\mathbf{u_1}$ and $\mathbf{w}$ is positive; a rigorous proof is given in the commentaries. Since $\mathbf{v} + \mathbf{w}$ is a multiple of $\mathbf{u_2}$ and $\mathbf{u_2}$ is orthogonal to $\mathbf{u_1}$, it follows that the inner product of $\mathbf{v}$ and $\mathbf{u_1}$ must be negative (note that the inner product of $\mathbf{v} + \mathbf{w}$ and $\mathbf{u_1}$ is zero). Also, since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent it follows that their inner product lies in the open interval $(-1, 1)$, and this in turn implies that the inner product of $\mathbf{u_2}$ (a positive multiple of $\mathbf{v} + \mathbf{w}$) with both $\mathbf{v}$ and $\mathbf{w}$ must be nonzero because the inner products of $\mathbf{v}$ and $\mathbf{w}$ with $\mathbf{v} + \mathbf{w}$ are both equal to $1 + \langle \mathbf{v}, \mathbf{w} \rangle$.



Suppose now that $\mathbf{u_1}$ and $\mathbf{u_2}$ are the standard unit vectors in the coordinate plane. Then there is a homeomorphism of this plane to itself which maps the vertical axis to itself by the identity and flattens the angle $\mathbf{v}\,\mathbf{0}\,\mathbf{w}$ (where $\mathbf{0}$ is the origin) into the horizontal axis. This is illustrated by the drawing below, in which regions of the same color correspond under the homeomorphism.



As usual, it is necessary to express the homeomorphism formally. Actually, the inverse homeomorphism is easier to express in terms of equations, so we shall describe the latter instead. Given a point (x, y) in the coordinate plane, its image under the inverse is equal to $x\,\mathbf{w} + y\,\mathbf{u_2}$ if x is nonnegative and $|x|\,\mathbf{v} + y\,\mathbf{u_2}$ if x is nonpositive.