Background on function spaces

If X is a compact Hausdorff space and Y is a metric space, then one natural definition of a topology on the set $\mathcal{F}(X,Y)$ of continuous functions from X to Y is given by the **uniform metric**, which is defined by

$$\mathbf{d}_{\mathcal{F}}(f,g) \quad = \quad \max_{X\in X}\ \mathbf{d}_Y\big(\,f(x),\,g(x)\,\big)\ .$$

These spaces are discussed in Unit IV of the 205A notes.

For many reasons it is desirable to find similar definitions of topologies on sets of continuous functions $\mathcal{F}(X,Y)$ in more general situations where X is not necessarily compact and the topology on Y is not given in terms of a metric (at least explicitly). The appropriate generalization of the preceding definition is given by the *compact-open topology*. This construction is defined and studied in Section 46 of Munkres, and it is the smallest topology such that that the sets

$$\mathcal{W}(C,U) \quad = \quad \{\ f\in\mathcal{F}(X,Y)\ |\ f[C]\ \subset U\ ,$$

$$\text{where } C\subset X \text{ is compact and } U\subset Y \text{ is open }\}$$

are open (hence the displayed sets form a subbase for the compact-open topology; by construction $\mathcal{W}(C,U)$ is the set of all continuous functions sending C to U).

Theorem 46.8 in Munkres implies that the compact-open topology and the topology generated by the uniform metric are the same if X is a compact Hausdorff space and Y is a metric space. More generally, if X is locally compact Hausdorff then Munkres defines a topology for uniform convergence on compact subsets (this notion is particularly significant in the theory of functions of a complex variable), and it turns out that for such choices of X the compact open topology coincides with the topology that one would expect from the topology for uniform convergence on compact subsets.

Finally, we shall note that if U is open in some $\mathbf{R}^m$ and Y is metric, then the topology of uniform convergence on compact subsets can be defined using a metric as follows: Express U as an increasing union of compact subsets C_n such that the union of their interiors is also U (possible by second countability and local compactness), for each pair of continuous functions $f,g:U\to Y$ let

$$\rho_n(f,g) \quad = \quad \max_{x\in C_n}\ \mathbf{d}\big(\,f(x),\,g(x)\,\big)$$

and set

$$\rho \quad = \quad \sum_{n=0}^{\infty}\ \frac{\rho_n}{2^n(1+\rho_n)}\ .$$

It is then a straightforward exercise to show that ρ defines a metric on $\mathcal{F}(X,Y)$ and the metric topology determined by ρ is the topology of uniform convergence on compact subsets. [The proof of the latter is related to a standard exercise on metric spaces; namely, if $\mathbf{d}$ is the given metric on a metric space, then $\widetilde{\mathbf{d}}\ =\ \mathbf{d}/(1+\mathbf{d})$ is a another metric defining the same open sets, and in this new metric the distances between pairs of points are always less than 1.]