Comments on the Lifting Criterion

A basic step in the proof of the Lifting Criterion in Section 79 of Munkres is the following observation:

> *Suppose that $f : (Y, y_0) \to (B, b_0)$ is a continuous map of arcwise connected, locally arcwise connected spaces, and let $p : (E, e_0) \to (B, b_0)$ be a base point preserving covering space projection such that E is also connected and the image of the assoicated map of fundamental groups f_* is contained in the image of p_*. Let α and β be continuous curves joining y_0 to $y \in Y$, and let $\widetilde{f\alpha}$ and $\widetilde{f\beta}$ denote the unique liftings of $f \circ \alpha$ and $f \circ \beta$ to continuous curves in E whose values at 0 are e_0. Then $\widetilde{f\alpha}(1) = \widetilde{f\beta}(1)$.*

Proof of assertion. Let $\varphi = \alpha + (-\beta)$, which is a closed curve based at y_0. Then $f \circ \varphi$ is a closed curve in B, and the fundamental group condition plus the Covering Homotopy Property show that the unique lifting $\gamma = \widetilde{\varphi}$ of φ to a curve in E with initial condition e_0 is also a closed curve. It follows immediately that the curve $\gamma_1(t) = \gamma\left(\frac{1}{2}t\right)$ (for $t \in [0,1]$) is a lifting of $f \circ \alpha$ to E with initial condition e_0 and $\gamma_2(t) = \gamma\left(1 - \frac{1}{2}\,t\right)$ (for $t \in [0,1]$) is a lifting of $f \circ \beta$ to E with initial condition e_0. Therefore $\gamma_1 = \widetilde{f\alpha}$ and $\gamma_2 = \widetilde{f\beta}$. By these formulas, the values of these curves at $t = 1$ are given by $\gamma_1(1) = \gamma(\frac{1}{2})$ and $\gamma_2(1) = \gamma(\frac{1}{2})$ respectively, and the conclusion of the assertion follows directly from these equations.■