CLARIFICATIONS TO COMMENTARIES

PROOF OF THE SEIFERT-VAN KAMPEN THEOREM. (pp. 48–49) Here are some additional details and modifications. We begin with material in the final two paragraphs of page 48:

> *We claim that E is arcwise connected ... by construction, if e_1 and e_2 are two points in F such that $g \cdot e_1 = e_2$ for some g in the image of the fundamental group of U, it follows that e_1 can be joined to e_2 by a continuous curve whose image lies in the inverse image of U in E; a similar conclusion holds if we replace U be V in the preceding statement.*

It is easier to prove the connectedness of E if we modify the preceding assertion as follows: *Suppose that e_0 is the base point of E and $g \in \Gamma$, and let $h \in \Gamma$ be an element which lies in the image of either $\pi_1(U)$ or $\pi_1(V)$. Then $g\,e_0$ and $g\,h\,e_0$ lie in the same component of E.* — Given this, one can use the fact that the images of $\pi_1(U)$ and $\pi_1(V)$ generate Γ to conclude that every point in F lies in the same component as e_0 and hence E is connected. Specifically, if we write $g = h_1 \ \cdots \ h_k$ for h_i satisfying the given conditions and lets g_0 denote the product of the first i factors for $0 \leq i \leq k$ (with $g_0 = 1$), then by induction we have that each $g_i \cdot e_0$ lies in the same component of E as e_0.

We shall only consider the case where h comes from the fundamental group of U; the other case follows by systematically replacing U with V throughout the discussion. It will help to have some notation. Let $k_U : \widetilde{U} \to E$ be the inclusion map given by the construction of U, and let u_0 denote the base point of $\widetilde{U}$, so that k_U maps u_0 to e_0. Suppose that $h \in \Gamma$ lies in the image of $\pi_1(U)$, and let h' map to h. By construction we know that k_u sends $h'\,u_0$ to $h\,e_0$. Let η be the curve in $\widetilde{U}$ joining u_0 to $h'\,u_0$. Then it follows that $k_U \circ \eta$ joins e_0 to $h\,e_0$, proving the assertion when h comes from $\pi_1(U)$; as noted before, a similar argument applies if h comes from $\pi_1(V)$, and by the remarks in the preceding paragraph it follows that E is connected as required.■

Next, we shall examine the following statements from page 49 more closely:

> *[We have] the diagram of morphisms displayed below, in which the square is commutative (all compositions of morphisms between two objects in this part of the diagram are equal).*

$$
\begin{array}{ccccccc}
\pi_1(U \cap V) & \longrightarrow & \pi_1(U \cap V) & & & & \\
\big\downarrow & & \big\downarrow J(U) & & & & \\
\pi_1(U \cap V) & \xrightarrow{J(V)} & \Gamma & \xrightarrow{\Phi} & \pi_x(X) & \xrightarrow{\partial} & \Gamma
\end{array}
$$

> *The map Φ is the homomorphism given by the universal mapping property of the pushout group Γ (see the commentary to Section 70). If we can show that $\partial \circ \Phi$ is the identity, then it will follow that Φ is injective. Since we already know that Φ is surjective (see Section 70), it will follow that Φ is an isomorphism, and the proof will be complete.*

In the subsequent discussion on page 49, the key point is to prove that the composites

$$\pi_1(U) \longrightarrow P \longrightarrow \pi_1(X) \longrightarrow \Gamma \qquad\qquad \pi_1(V) \longrightarrow P \longrightarrow \pi_1(X) \longrightarrow \Gamma$$

are just the standard maps $J(U)$ and $J(V)$ from $\pi_1(U)$ and $\pi_1(V)$ into the pushout Γ. It will be helpful to let i_{U*} and i_{V*} denote the maps of fundamental groups induced by the inclusions of U and V in X; by construction we have $i_{U*} = \Phi \circ J(U)$ and $i_{V*} = \Phi \circ J(V)$.

As before, it suffices to show that $\partial \circ \Phi \circ J(U) = J(U)$, for the argument in the other case will follow by systematic substitution of V for U throughout. — Let h' be an element in $\pi_1(U)$, and let h be its image in Γ. By construction, the covering space transformation determined by $\partial \circ \Phi(h) \in \Gamma$ sends the base point e_0 to $\Phi(h) \cdot e_0 = i_{U*}(h') \cdot e_0$. On the other hand, we also know that the covering space transformation of $\widetilde{U}$ associated to h' sends u_0 to $h' \cdot u_0$, and if we apply the mapping k_U from the previous discussion, it follows that the covering space transformation of E associated to $J(U)(h')$ sends $e_0 = k_U(u_0)$ to $i_{U*}(h') \cdot e_0$.

The preceding argument shows that $\partial \circ i_{U*} = J(U)$, and the identity in the first sentence of the preceding paragraph then follows because $i_{U*} = \Phi \circ J(U)$. As noted above, we have a similar identity involving V. Taken together, these imply that the restrictions of $\partial \circ \Phi$ to the images of $J(U)$ and $J(V)$ are the identity, and since these images generate Γ it follows that $\partial \circ \Phi$ must be the identity, as claimed.■