CLARIFICATIONS TO COMMENTARIES – II

Reduced paths and simple circuits. The definitions of these on page 50 of the commentaries were somewhat informal, and they should be amplified as follows.

Definition. Let $E_1\ ,\cdots\ ,E_n$ be an edge-path sequence such that the vertices of E_i are $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$. This sequence is said to be *reduced* if $\mathbf{v}_1\ ,\cdots\ ,\mathbf{v}_n$ are distinct and either $n \neq 2$ or else $\mathbf{v}_0 \neq \mathbf{v}_2$ (the latter case is just a sequence with $E_2 = E_1$, physically corresponding to going first along E_1 in one direction and then back in the opposite direction).

The following result is more or less predictable but still important.

PROPOSITION. *If two distinct vertices* $\mathbf{x}$ *and* $\mathbf{y}$ *can be connected by an edge-path sequence, then they can be connected by a reduced sequence.*

Proof. Take a sequence with a minimum number of edges. We claim it is automatically reduced. If not, then there is a first vertex which is repeated, and a first time at which it is repeated. In other words, there is a minimal pair (i, j) such that $i < j$ and $\mathbf{v}_i = \mathbf{v}_j$, which means that if (p, q) is any other pair with this property we have $p \geq i$ and $q > j$. If we remove E_{i+1} through E_j from the edge-path sequence, we obtain a shorter sequence which joins the given two vertices.■

Note that the converse is false. For example, take X to be the triangle graph in the plane whose vertices are the three noncollinear points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and whose edges are the three line segments joining these pairs of points. Then $\{\mathbf{ab}, \mathbf{bc}\}$ and $\{\,\mathbf{ac}\,\}$ are two reduced edge-path sequences joining $\mathbf{a}$ to $\mathbf{c}$ such that one consists of two edges and the other consists of one edge.

Another consequence of the definitions worth noting is that *every simple circuit in a graph contains at least three edges.*