CLARIFICATIONS TO COMMENTARIES – III

Recognition of trees.   The results of this course provide several simple criteria for recognizing when a connected graph is a tree.

THEOREM. *If X is a connected graph, then the following are equivalent:*
   (i) X is a tree.
   (ii) X is contractible.
   (iii) X is simply connected.

Proof.   We already know that the first condition implies the second and the second implies the third, so it is only necessary to prove that (iii) implies (i). However, if T is a maximal tree in X and $T \neq X$, then we know that the fundamental group of X is a free group on k generators, where $k > 0$ is the number of edges which are in X but not in T. Therefore if X is simply connected we must have $T = X$.∎

COROLLARY.   *A connected graph X is a tree if and only if $\chi(X) = 1$.*

Proof.   We know that $\chi(X) = 1$ if and only if X is simply connected.∎

   REMARK. More generally, one has the following criteria for recognizing whether two connected graphs X and Y are homotopy equivalent:

   (1) *The connected graphs X and Y are homotopy equivalent if and only if their fundamental groups are isomorphic.*∎

   (2) *The connected graphs X and Y are homotopy equivalent if and only if their Euler characteristics are equal.*∎

   The results of this course show that the fundamental groups are isomorphic if and only if the Euler characteristics are equal, so (2) will follow from (1). Proving the latter is not all that difficult, but we shall not give the details here.