EXERCISES FOR MATHEMATICS 205B

WINTER 2008

Full information on the references for the course (Munkres and Hatcher) appear in the file `math205Bcommentaries.pdf`. Problems marked with * are usually somewhat challenging and should be regarded as optional.

Munkres, Section 51

Munkres, p. 330: 1–3

Additional exercises

1. Let X be a topological space, and let P be a topological space consisting of exactly one point (it has a unique topology). Explain why the set of homotopy classes $[P, X]$ is in 1–1 correspondence with the set of arc components of X.

2. Let Y be a nonempty topological space with the indiscrete topology (*i.e.*, $\emptyset$ and Y are the only open sets), and let X be an arbitrary nonempty topological space. Prove that $[X, Y]$ consists of a single point. [*Hint:* For all topological spaces W, every map of sets from W to Y is continuous. Using this, show that if $A \subset B$ is a subspace and $g : A \to Y$ is continuous, then g extends to a continuous map from B to Y.]

3. Let Y be a nonempty space with the discrete topology (all subsets are open), and let X be a nonempty connected space. Prove that there is a 1–1 correspondence between $[X, Y]$ and Y.

Munkres, Section 52

Munkres, pp. 334–335: 1–7

Additional exercises

1. Let X be the Cantor Set (see Munkres, p. 178), and let P be a space consisting of a single point. Prove that $[P, Y]$ is uncountable, and using this explain why X does not have the homotopy type of an open subset in some Euclidean space $\mathbf{R}^n$. [*Hint:* By construction X is an intersection of a decreasing sequence of compact sets X_n, where each X_n is a union of 2^n pairwise disjoint closed intervals, each of which has length $1/3^n$. If γ is a continuous map from $[0, 1]$ to X_n, why does this imply that $|\gamma(t) - \gamma(0)| < 3^{-n}$ for all t? What does this imply for continuous maps from $[0, 1]$ to X and for the set of homotopy classes $[P, X]$? Finally, if a space Y has the homotopy type of an open set in some $\mathbf{R}^n$, what can one say about the cardinality of $[P, Y]$? Combine the answers of the last two questions to complete the argument.]

2. Let U be an open subset of $\mathbf{R}^n$. Modify the argument in the commentary for § 51 of Munkres to show that $\pi_1(U, u_0)$ is at most countable, where $u_0 \in U$ is arbitrary.

3. Suppose that X, Y and Z are nonempty topological spaces and that $p : Y \times Z \to Y$ and $q : Y \times Z \to Z$ are the projections onto the two factors. Prove that the map $\varphi : [X, Y \times Z] \to [X, Y] \times [X, Z]$ which sends a class $u = [f]$ to $([p \circ f], [q \circ f])$ is 1–1 and onto.

4. Suppose that X is a nonempty compact topological space and Y is an increasing union $\cup_n C_n$ of a sequence of compact subspaces such that every compact subset of Y is contained in some subset C_n. Let $j_n : C_n \to Y$ be the inclusion, and for $n \le m$ let $i_{n,m} : C_n \to C_m$ denote the inclusion (hence $j_n = j_m \circ i_{n,m}$ for all n and m).

(i) Prove that every nonempty open subset in $\mathbf{R}^n$ is a describable as such a countable union of compact subsets. [*Hint:* Use the Lindelöf Property.]

(ii) Explain why every class $u \in [X, Y]$ has the form $(j_n)_*(v)$ for some n and some $v \in [X, C_n]$.

(iii) Suppose we are given v_0 and v_1 in $[X, C_n]$ such that $(j_n)_*(v_0) = (j_n)_*(v_1)$. Prove that there is some $m \ge n$ such that $(i_{n,m})_*(v_0) = (i_{n,m})_*(v_1)$.

5. Let U be an open subset of $\mathbf{R}^n$. A **broken line curve** is a continuous curve $\gamma : [a, b] \to U$ such that the following holds: There is a partition of $[a, b]$ given by

$$a = x_0 < x_1 < \cdots < x_k = b$$

such that the restriction of γ to each closed subinterval $[x_{i-1}, x_i]$ is a straight line segment which has a parametrization of the form

$$\beta(t) = \left(\frac{x_{i-1} - t}{\Delta_i}\right) \cdot \gamma(x_{i-1}) + \left(\frac{t - x_i}{\Delta_i}\right) \cdot \gamma(x_i)$$

where $\Delta_i = x_i - x_{i-1}$. — Prove that every element of $\pi_1(U, u_0)$ (where $u_0 \in U$) is representable by a broken line curve.

6. Let X be a topological space, let $x_0 \in X$, and let X_0 denote the path component of x_0. Explain why $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X_0, x_0)$.

7.* Let X be an arcwise connected space, let $x_0 \in X$, and let $F; \pi_1(X, x_0) \to [S^1, X]$ be the map which takes the base point preserving homotopy class of a closed curve γ to the ordinary homotopy class (the *free homotopy class*) of γ viewed as a function from S^1 to X.

(i) Prove that F is onto. [*Hint:* Let α be a closed curve whose initial and final values are given by x_1, and let γ be a closed curve joining x_0 to x_1. Then $(\gamma + \alpha) + (-\gamma)$ is a closed curve whose initial and final values are given by x_0. Prove that α is freely homotopic to $(\gamma + \alpha) + (-\gamma)$.]

(ii) Prove that $F(g_1) = F(g_2)$ if and only if there is some $h \in \pi_1(X, x_0)$ such that $g_2 = h\, g_1\, h^{-1}$. [*Hint:* If $F(g_1) = F(g_2)$ then one has a homotopy from $[0,1] \times [0,1]$ to X such that the restrictions to $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ represent g_1 and g_2 and the restrictions to $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ are the same curve. The initial and final values of this curve are x_0 so it represents an element h in the fundamental group.]

8.* Let $n \ge 2$, and let $x_0 \in S^n$. Prove that $\pi_1(S^n, x_0)$ is trivial. [*Hint:* Let $i : S^n \to \mathbf{R}^{n+1} - \{\mathbf{0}\}$ denote the inclusion mapping, and let $r : \mathbf{R}^{n+1} - \{\mathbf{0}\}$ denote the retraction map sending $\mathbf{v}$ to $|\mathbf{v}|^{-1} \cdot \mathbf{v} \in S^n$. Both of these maps are base point preserving, and the composite $r \circ i$ is the identity. Using this, show that it suffices to prove that if γ is a closed curve in S^n, then $i \circ \gamma$ is homotopic to a constant map. By the preceding exercise the latter curve is base point preservingly homotopic to a broken line curve. Explain why the image of such a map is contained in a finite union of 2-dimensional vector subspaces in $\mathbf{R}^{n+1}$. One can now apply theorems in linear algebra or 205A to show that this finite union is a proper subset, and in fact one can find a nonzero vector $\mathbf{w}$ such that all nonzero multiples of $\mathbf{w}$ lie in the complement of this set. If β is the broken line curve, this implies that the image of $r \circ \beta$ is contained in the complement of a point in S^n. Using this, show that the class $[\gamma] = [r \circ \beta]$ in $\pi_1(S^n, x)$ lies in the image of $\pi_1(S^n - \{w\}, x_0)$ for some w. Using the fact that $S^n - \{w_0\}$ is homeomorphic to $\mathbf{R}^n$, explain why the image of $\pi_1(S^n - \{w\}, x_0)$ in $\pi_1(S^n, x_0)$ must be trivial.]

9.* (*This exercise involves concepts from category theory that are not officially part of the course, and as such it is even more optional than most such exercises.*)

Let X be an arcwise connected topological space. For each pair of points $a, b \in X$ define $\prod X(a,b)$ to be the set of **endpoint-preserving** path homotopy classes of curves from a to b, and for each triple of points a, b, c define a binary operation

$$\Phi_{a,b,c} : \prod X(a,b) \times \prod X(b,c) \to \prod X(a,c)$$

by $\Phi([\alpha], [\beta]) = [\alpha + \beta]$.

(i) Explain why Φ is well-defined and $\prod X(a,a)$ is the fundamental group of (X, a).

(ii) Show that $(X, \prod X, \Phi)$ is a category in which all morphisms in each set $\prod X(a,b)$ are isomorphisms ($i.e.$, a *groupoid*). This category is known as the **fundamental groupoid** of X, and in some sense it is the ultimate "basepoint free" version of the fundamental group. [*Hint:* Show that the constant classes $[C_a]$ behave like identities and that $[-\alpha]$ is an inverse to $[\alpha]$. Versions of these facts, formulated without the category-theoretic language, were used in the proof that the isomorphism type of fundamental group for an arcwise connected space does not depend upon the choice of base point.]

Munkres, Section 53

Munkres, pp. 341: 1–6

Additional exercises

1. Suppose we are given covering maps $p_i : E_i \to X$ $(i = 1, 2)$ and a factorization $p : E_1 \to E_2$ such that $p_1 = p_2 \circ p$. Prove that p is also a covering map.

2. Let $p : E \to X$ be a covering map, and let $f : Y \to X$ be continuous. Define the *pullback*

$$Y \times_X E := \{(e, y) \in Y \times E |\ f(y) = p(e)\}.$$

Let $p_{(Y,f)} = \mathrm{proj}_Y | Y \times_X E$.

(i) Prove that $p_{(Y,f)}$ is a covering map. Also prove that f lifts to E if and only if there is a map $s : Y \to Y \times_X E$ such that $p_{(Y,f)} s = 1_Y$.

(ii) Suppose also that f is the inclusion of a subspace. Prove that there is a homeomorphism $h : Y \times_X E \to p^{-1}(Y)$ such that $p \circ h = p_{(Y,f)}$.

NOTATION. If the condition in (ii) holds we sometimes denote the covering space over Y by $E|Y$ (in words, E restricted to Y).

3. Suppose that $p : E \to X$ is a covering space projection and X is totally disconnected ($i.e.$, the topology has a base of sets that are both open and closed). Prove that E is also totally disconnected.

Munkres, Section 54

Munkres, pp. 347–348: 1–8

Additional exercises

1. Let T^n be the product of $n \geq 3$ copies of S^1 with itself, and let $e \in T^n$ be the point $(1, \cdots, 1)$. Prove that $\pi_1(T^n, e)$ is isomorphic to a direct product of n copies of $\mathbf{Z}$ and explain why there is a covering space projection from $\mathbf{R}^n$ to T^n. [*Hint:* The result is also true if $n = 1$ or 2, but these have already been shown.]

2. If f is a base point preserving continuous map from (T^n, e) to itself, then by the preceding result we know that the induced map f_* of fundamental groups is given by an $n \times n$ matrix with integral entries. If f is a homotopy equivalence of pointed spaces, show that this matrix must have determinant ± 1. Conversely, show that every such matrix can be realized by a base point preserving homeomorphism from T^n to itself. [*Hint:* For the second part, if A is an arbitrary $n \times n$ matrix with integral entries and $p : \mathbf{R}^n \to T^n$ is the map sending $(t_1, \cdots, t_n)$ to $(e^{2\pi \mathbf{i}\, t_1}, \cdots, e^{2\pi \mathbf{i}\, t_n})$, prove that there is a unique base point preserving map f_A from T^n to itself such that $f_A \circ p(\mathbf{x}) = p(A\mathbf{x})$, where $\mathbf{x} = (t_1, \cdots, t_n)$.]

Munkres, Section 55

Munkres, p. 353: 1–3

Hatcher, pp. 18–20: 3, 10
Hatcher, pp. 38–40: 11, 13, 17 (but replace $S^1 \vee S^1$ with $S^1 \times S^1$), 20

Additional exercises

1. Suppose we have $A \subset B \subset X$ such that A is a retract of B and B is a retract of X. Prove that A is a retract of X.

2. Suppose that A is a retract of X; let $j : A \to X$ be the inclusion mapping, and let $x_0 \in A$. If H is the image of the fundamental group of A under the mapping h_*, prove that there is a normal subgroup K of $\pi_1(X, x_0)$ such that the latter is generated by H and K, and we have $H \cap K = \{1\}$. [*Hint:* Let $r : X \to A$ be the associated retraction, and consider the kernel of r_*.]

3. Let B be the set constructed at the beginning of the proof for Corollary 55.7 on pp. 351–352. Prove that B is homeomorphic to D^2. [*Hint:* Show that one can define a homeomorphism from B to the solid circle quadrant

$$Q = \{\, (x, y) \in \mathbf{R}^2 \mid x \ \geq 0 \,, \quad y \ \geq \ 0 \,, \quad x^2 + y^2 \leq 1 \,\}$$

by projecting down to the first two coordinates. Next, show that this quadrant is homeomorphic to the closed semicircular region

$$Q = \{\, (x, y) \in \mathbf{R}^2 \mid x \ \geq 0 \,, \quad x^2 + y^2 \leq 1 \,\}$$

by the map sending (x, y) to $(x^2 - y^2,\, 2xy)$; in terms of complex numbers this is just the map sending z to z^2. Finally, show that H is homeomorphic to the closed disk E with radius $\frac{1}{2}$ and center $\mathbf{z} = (0, \frac{1}{2})$; start by checking that $E \subset H$ and E meets the boundary of H in the points $(0, 0)$ and $(0, 1)$. Here is a sketch of the construction for the homeomorphism: The center goes to itself, and for every other point $\mathbf{y}$ the image of $\mathbf{y}$ will be a point $\mathbf{z} + u(\mathbf{z} - \mathbf{y})$ for a suitably construtcted $u > 0$ (in other words, on the ray beginning at $\mathbf{z}$ and passing through $\mathbf{y}$. Specifically, there is a unique $s(\mathbf{y}, \mathbf{z}) > 0$ (in fact ≥ 1) such that the given ray meets the boundary of H at

$$\mathbf{z} \;+\; s(\mathbf{y}, \mathbf{z}) \cdot (\mathbf{z} - \mathbf{y}) \;.$$

There are two formulas for s, depending upon whether the intersection point lies on the upper or lower curve of H; show that they define a continuous function for all $\mathbf{y} \neq \mathbf{z}$ in H. It is then necessary to show that the function sending $\mathbf{y}$ to

$$2 \cdot s(\mathbf{x},\, \mathbf{y}) \cdot \big(\mathbf{y} - \mathbf{z}\big)$$

is continuous and 1–1 on $H - \{\mathbf{z}\}$, and if we extend this to all of H by sending $\mathbf{z}$ to itself, then the resulting extension is also continuous and 1–1.

COMMENT. At the top of page 352 in Munkres, there is a remark that the conclusion of the exercise "is easy to see." This may be true intuitively, but often such phrases can be used to avoid writing down messy details, and it is usually appropriate to be at least somewhat skeptical of such claims.

Munkres, Section 56

Additional exercise

1. Suppose that we are given a circle in the complex plane which is defined by the equation $|z - a| = r$ and parametrized in the counterclockwise sense, let $z_0 = a + r$, and let b be a point which does not lie on this circle. Let $x(b)$ denote the class of this circle in $\pi_1(\mathbf{C} - \{b\}, z_0) \cong \mathbf{Z}$. Prove that $x(b)$ is a generator of the fundamental group if $|b - a| < r$ (so that b lies inside the circle) and $x(b)$ is trivial if $|b - a| > r$ (so that b lies outside the circle). [*Hints:* In the first case let γ be a small counterclockwise circle centered at b and show that the original circle is homotopic to γ. In the second case explain why the disk bounded by the original curve lies in the complement of $\{b\}$.]

Munkres, Section 58

Munkres, p. 366: 1–3, 5–7, 9

Hatcher, pp. 18–20: 1, 2, 4, 5, 9, 13

Additional exercises

1. Prove that if $f : X \to Y$ is continuous and X and Y are Hausdorff, then the mapping cylinder $M(f)$ is also Hausdorff. Also prove the same result for the pointed mapping cylinder if f is base point preserving. [*Hint:* Consider the unpointed case, and suppose that we are given distinct equivalence classes u and v such that u contains no points of Y. It follows that v consists of exactly one point (x, t) such that $t < 1$. Why are there disjoint open subsets of $X \times [0,1] \amalg Y$? containing (x, t) and $X \times \{1\} \amalg Y$? Next suppose that both equivalence classes contain a point of Y; why must each contain exactly one such point? Given the points y_u and y_v as in the previous sentence, take disjoint neighborhoods of them, and show that there are disjoint open subsets of their images in $X \times [0,1] \amalg Y$ such that these open sets are unions of equivalence classes. Finally, explain how the preceding can be carried out to prove the Hausdorff property in the pointed case.]

2. Prove the analog of the mapping cylinder proposition (in the commentary) for base point preserving maps.

3.* As noted in the next to last paragraph on page 358 of Munkres, the Figure 8 and Figure Theta spaces have the same homotopy type, but neither is a deformation retract of the other, and in fact neither is homeomorphic to a subspace of the other. Prove the last assertion in the preceding sentence. [*Hint:* For both spaces, define a nonsingular point y to be a point such that y has a neighborhood base of open sets V such that $V - \{y\}$ has exactly 2 components. How many singular points does each space have? For each singular point z of the Figure 8, explain why z has a neighborhood base of open sets V such that $V - \{y\}$ has exactly 4 components but there is no similar neighborhood base for which the deleted neighborhoods all have fewer components, and prove that every singular point of the Figure Theta has a neighborhood base of open sets V such that $V - \{y\}$ has exactly 3 components but there is no similar neighborhood base for which the deleted neighborhoods all have fewer components. Suppose now that we have a continuous 1–1 map h from one of these spaces to the other. Explain why a singular point cannot go to a nonsingular point, and if we have a singular point w such that every open neighborhood U of has a subneighborhood V such that $V - \{w\}$ has n components, then every neighborhood U' of $h(w)$ must have a subneighborhood V' such that $V' - \{\, h(w) \,\}$ has at least n components.]

4. As in the hint for the preceding exercise, for a noteworthy family of spaces (which includes the usual numerals and letters of the alphabet), at each point x one can define a *local index*, which is an integer n such that x has a neighborhood base $\{V_\alpha \mid \alpha \in A\}$ such that each deleted neighborhood $V_\alpha - \{x\}$ has exactly n components, and there is no neighborhood base such that the associated deleted neighborhoods have fewer components. Using this notion, show that there are at least **7** homeomorphism types represented by the standard hexadecimal digits as written below (in sans-serif type):

0 1 2 3 4 5 6 7 8 9 A B C D E F

Are new homeomorphism types added if we consider the remaining letters of the alphabet? Explain.

Munkres, Section 59

Munkres, p. 370: 1–4

Hatcher, pp. 38–40: 18(a)
Hatcher, pp. 52–55: 2, 8

Additional exercises

1. Let X be a bouquet of three circles defined formally as

$$S^1 \vee S^1 \vee S^1 \;=\; S^1 \times \{1\} \times \{1\} \;\cup\; \{1\} \times S^1 \times \{1\} \;\cup\; \{1\} \times S^1 \times \{1\} \;\subset\; T^3$$

and let $j : S^1 \vee S^1 \vee S^1 \to T^3$ be the inclusion map. Prove that j_* defines a surjection in fundamental groups, and using this show that $\pi_1(S^1 \vee S^1 \vee S^1, e)$ cannot be generated by two elements. [*Hint:* Let G be a subgroup of $\pi_1(S^1 \vee S^1 \vee S^1, e)$ which is generated by two elements, and explain why $j_*[G]$ is a proper subgroup of $\pi_1(T^3, e)$.]

2. Prove that $\pi_1(S^1 \vee S^1 \vee S^1, e)$ is not abelian. [*Hint:* Prove that $S^1 \vee S^1$ is a retract of $S^1 \vee S^1 \vee S^1$. Also, note that if G is abelian then so is every subgroup of G.]

Munkres, Section 60

Munkres, p. 375: 1, 5

Additional exercise

1.* Let X be the a point union of S^1 and $\mathbf{RP}^2$. Show that X has a 2-sheeted covering which is homeomorphic to $S^2 \cup A \cup B$, where A and B are homeomorphic to S^1 such that $A \cap B = \emptyset$ and there is some $\mathbf{e} \in S^2$ such that $A \cap S^2 = \{\mathbf{e}\}$ and $A \cap S^2 = \{-\mathbf{e}\}$. Prove that the fundamental group of $S^2 \cup A \cup B$ maps onto the fundamental group of $S^1 \vee S^1$, and use this to conclude that the fundamental group of X must be nonabelian. [*Hint:* Use Theorem 54.6 in Munkres for the final part of the problem.]

Munkres, Section 61

Munkres, p. 380–381: 1–2

Additional exercises

1. Suppose that A is a compact subset of $\mathbf{R}^n$, where $n \geq 2$. Prove that there is a 1–1 correspondence between the components of $\mathbf{R}^n - A$ and the components of $S^n - A$ such that (*i*) all but one components of these two sets are equal, (*ii*) in the exceptional case, the components C_1 and C_2 of the respective spaces are related by the equation $C_2 = C_1 \cup \{\infty\}$. [*Hint:* Why does $\mathbf{R}^n - A$ contain some set of the form $\{ \mathbf{x} \mid |\mathbf{x}| > M \}$?].

2.* Using the methods in the commentary for this section, prove that a piecewise smooth simple arc in $\mathbf{R}$ is locally flat; the proof should be a combination of the arguments given in the commentary for smooth curves and for broken lines.

3. Suppose that Γ_1 and Γ_2 are the images of locally flat simple closed curves in $\mathbf{R}^2$, and suppose that each component of Γ_2 contains a point of Γ_1. Prove that $\Gamma_1 \cap \Gamma_2$ contains at least two points. Give examples where one curve is a circle, the other is an ellipse, and the number of intersection points is 3 or 4. Finally, give rough sketches of examples for which the number of intersection points is an arbitrary integer ≥ 5 (you do not have to give formal proofs for this part of the problem).

Munkres, Section 63

Munkres, p. 384–385: 5
Munkres, p. 393–394: 1

(Assume curves are locally flat if it seems helpful to do so.)

Munkres, Section 64

Munkres, p. 398: 1(a)

Additional exercises

1. Prove that the Figure 8 space is homeomorphic to the underlying topological space of a linear graph (in the sense of Munkres).

2.* Suppose that we are given a linear graph structure on the Figure 8 space as in the preceding exercise, and suppose that $F \subset \mathbf{R}^2$ is a locally tame subset which is homeomorphic to the Figure 8 space. Write $F = A \cup B$ where A and B are simple closed curves and their intersection is a single point. Prove that $\mathbf{R}^2 - F$ has three components U, V, W such that the set of limit points for one component is A, the set of limit points for another component is B, and the set of points for the remaining component is $A \cup B$.

Munkres, Section 68

Munkres, p. 421: 2, 3

Additional exercise

1. Given three groups G, H and K, use the Universal Mapping Property to show that iterated free products $(G * H) * K$ and $G * (H * K)$ are both isomorphic to the threefold free product $G * H * K$. Similarly, show that $G * H$ is isomorphic to $H * G$.

Munkres, Section 69

Munkres, p. 425: 1, 3, 4

Additional exercises

1. Suppose we are given groups F_1 and F_2 with subsets $X_i \subset F_i$ such that F_i is a free group on X_i for $i = 1, 2$. Using the Universal Mapping Property, prove that $F_1 * F_2$ is free on the disjoint union $X_1 \amalg X_2$.

2. Give examples of groups H_1, H_2 and K such that H_1 is not isomorphic to H_2 but $H_1 * K$ is isomorphic to $H_2 * K$. [*Hint:* Let K be a free group on infinitely many generators, let H_1 be finite, and let $H_2 = H_1 * K$.]

Munkres, Section 70

Munkres, p. 433: 1, 3

Additional exercises

1. Suppose we are given a finitely presented group G with generators x and y and relations $x^3\,y^{-2}$, $x^2\,y\,x^{-2}\,y^{-1}$ and $y^3\,x\,y^{-3}\,x^{-1}$ (hence $x^3 = y^2$, and this element lies in the center because it commutes with a set of generators).
(i) If $[G, G]$ is the commutator subgroup of G, show that the abelianization $G/[G, G]$ is infinite cyclic.
(ii) Let N be the normal subgroup which is normally generated by $x\,y^{-1}$. Prove that $N = G$. [*Hint:* Compare the images of x and y in G/N and recall that $x^3 = y^2$.]

2. Suppose that the topological space X is the union of the arcwise connected open subspaces U and V such that $U \cap V$ is (nonempty and) arcwise connected, where all these spaces have the same base point. Assume further that the associated map of fundamental groups from $\pi_1(U \cap V)$ to $\pi_1(U)$ is onto and the associated map of fundamental groups from $\pi_1(U \cap V)$ to $\pi_1(V)$ is an isomorphism. Prove that the associated map from $\pi_1(U)$ to $\pi_1(X)$ is also an isomorphism.

Munkres, Section 71

Munkres, p. 438: 4, 5
Hatcher, pp. 52–55: 20

Additional exercises

1. Let $X \subset T^2$ be the union of the three circles $S^1 \times \{1\}$, $\{1\} \times S^1$ and $\{-1\} \times S^1$. If $q : X \to S^1 \vee S^1$ is the map which sends $(z, 1)$ to $(z^2, 1)$ and (ε, w) to $(1, w)$ where $\varepsilon = \pm 1$, verify that q is a two sheeted covering space projection and the associated map of fundamental groups defines a 1–1 homomorphism from a free group on three generators to a free group on two generators.

2. Let F_2 denote the free group on the generators x and y. Prove that there is a chain of subgroups

$$\cdots \subset H_n \subset H_{n-1} \subset \cdots \subset H_3 \subset F_2$$

such that for each $k \geq 3$ the subgroup H_k is free on k generators.

> REMARK. This contrasts sharply with the situation for *free abelian groups*, were every subgroup of a free abelian group on n generators is free abelian on m generators for some nonnegative integer $m \leq n$. At the end of this course we shall prove that F_2 even contains a subgroup which is free on a countably infinite set of generators.

Munkres, Section 72

Munkres, p. 441: 1

Additional exercises

1. Suppose that X is the space obtained from S^1 by attaching a 2-cell using the map z^2 from S^1 to itself. Prove that X is homeomorphic to $\mathbf{RP}^2$. [*Hint.* Consider the map from D^2 to the projective plane which first maps D^2 to the graph of the function $\sqrt{1 - |\mathbf{v}|^2}$, which lies in S^2, and then follow this by the quotient map from S^2 to the projective plane. Next, consider the quotient of D^2 such that two points are equivalent if and only if their images under this map are equal, and show that it is homeomorphic to X.]

2. The following exercise will prove the existence of an arcwise connected space X whose fundamental group is the additive group $\mathbf{Q}$ of rational numbers.

(i) Define a sequence of spaces K_n recursively as follows: Let M_n denote the mapping cylinder of the function z^n from S^1 to itself, let X_n correspond to the "top" part of the mapping cylinder (the domain of the map) and let Y_n correspond to the "bottom" (the codomain). Take the quotient *telescope space* T_∞ of this sequence to be the quotient of $\amalg_m M_n$ with Y_n identified to X_{n+1} for all $n \geq 1$. Let T_N be the union of the images of the first N mapping cylinders. Show that each T_n is homotopy equivalent to the circle, and show that the algebraic map determined by

$$\mathbf{Z} \cong \pi_1(X_0) \longrightarrow \pi_1(T_N) \cong \mathbf{Z}$$

is multiplication by $N!$ (= factorial).

(ii) Explain why the sets of the form $T_n - Y_n$ are open for all $n > 0$, use this to show that every compact set of T_∞ is contained in some T_m, and conclude the argument by using earlier exercises to show that the fundamental group of T_∞ is isomorphic to the additive group of rational numbers $\mathbf{Q}$. The homomorphism from the fundamental group of T_m to $\mathbf{Q}$ should take the generator of $\pi_1(T_m)$ to $1/m!$ for each m.

Munkres, Section 79

Munkres, pp. 483–484: 1, 2(a), 5(b)

Additional exercises

1. Let $p : E \to B$ be a covering space projection, where E and B are Hausdorff (as usual), connected and locally arcwise connected. Suppose that there is a *cross-section* to p; in other words, there is a continuous (base point preserving) map $\sigma : B \to E$ such that $p \circ \sigma$ is the identity. Prove that p is a homeomorphism. [*Hint:* Why must the map of fundamental groups p_* be surjective?]

2. Let $P \subset \mathbf{R}^2$ be the Polish circle. Prove that for each positive integer n there is a nontrivial connected covering space of P with n sheets, and prove similarly that there is a nontrivial connected covering space of P with infinitely many sheets. These examples show that the classification principle for (connected) covering spaces breaks down if the base space B is not locally arcwise connected, even if B itself is simply connected (which is the case if B is the Polish circle). [*Hint:* Imitate the constructions of the covering spaces of the circle.]

3. Suppose that $E \to X$ is a covering projection with countably many sheets, where X is connected, locally arcwise connected, semilocally simply connected, and separable metric. Prove that E is metrizable. [*Hint:* Apply Theorem 34.1 in Munkres.] — In particular, if U is an open subset of $\mathbf{R}^n$ and E is a connected covering space of U, then one can combine this with the countability result for $\pi_1(U)$ and the (still to be shown) existence of a simply connected covering space to prove that every (Hausdorff) covering space of U is metrizable.

4. Let $p : E \to X$ be a covering map (with the usual assumptions that all spaces be locally arcwise connected, but **not necessarily connected**), and let $f : A \to X$ be a subspace inclusion, where A and X are both connected and A is locally arcwise connected. Denote the pullback covering by $E|A$.

(i) Show that A is evenly covered if the induced map of fundamental groups f_* is the trivial homomorphism.

(ii) Show that if the induced map of fundamental groups f_* is onto, then $E|A$ is connected if E is connected.

(iii) Suppose that E is simply connected. Show that if the induced map of fundamental groups f_* is 1–1, then the components of $E|A$ are all simply connected.

Munkres, Section 80

Munkres, p. 487: 1(a)

Munkres, Section 81

Munkres, p. 492–494: 1(a), 5

Additional exercises

1. Determine the number of equivalence classes of based 2-sheeted covering spaces of $S^1 \vee S^1$, and determine the number of equivalence classes of regular based 4-sheeted coverings of the same space. [*Hints:* Every subgroup of index 2 is a normal subgroup, and normal subgroups of index n are the kernels of surjective homomorphisms onto groups of order n. Up to isomorphism there are only two groups of order 4.]

2. Let X be a Hausdorff space that is arcwise connected and semilocally simply connected, and let $f : X \to X$ be a homeomorphism which sends a base point x_0 into itself. The *mapping torus* of f is defined to be the quotient space M_f obtained from $X \times [0, 1]$ modulo the equivalence relation generated by $(x, 0) \sim (\, f(x),\, 1)$. We have already considered a nontrivial special case; namely, the Klein bottle (note that if f is the identity then the mapping torus is just $M \times S^1$).

(i) Let $\widetilde{X}$ denote the universal covering space of X, and let y_0 be a base point which projects to x_0. Prove that there is a unique base point preserving homeomorphism $\widetilde{f}$ from $\widetilde{X}$ to itself such that $q \circ \widetilde{f} = f \circ q$, where $q : \widetilde{X} \to X$ is the universal covering projection.

(ii) For each positive integer n show that M_{f^n} is an n-sheeted covering space of M_f, and also prove that $X \times \mathbf{R}$ is a covering space of M_f.

(iii) Let

$$\varphi : \mathbf{Z} \to \mathrm{Aut}\, \pi_1(X, x_0)$$

be the homomorphism $\varphi(n) = (f_*)^n\,[= (f^n)_*\,]$; Prove that $\pi_1(M_f, [x, 0])$ is the semidirect product of $\pi_1(X, x)$ and $\mathbf{Z}$ by φ. [*Hint:* Let $\widetilde{X}$ denote the univeral covering of X, and show that the universal covering space of M_f is homeomorphic to $\widetilde{X} \times \mathbf{R}$ wuch that the group of covering space automorphisms is generated by the group of all maps $T_g \times \mathrm{id}_{\mathbf{R}}$ and the covering space transformation

$$S_f(y, t) \quad = \quad \big(\,\widetilde{f}(y), t + 1\big)$$

where $\widetilde{f}$ is defined as in (i). Explain why the subgroup of all transformations of the form $T_g \times \mathrm{id}$ and the cyclic group generated by S_f intersect only in the identity element, and show that the composite $S_f \circ (T_g \times 1) \circ (S_f)^{-1}$ sends the base point $y_0 \in \widetilde{X}$ to $f \circ T_g(y_0)$.]

Munkres, Section 82

Munkres, p. 441: 1

Munkres, Section 84

Munkres, p. 513: 2

Additional exercises

1. Let $(X, \mathcal{E})$ be an edge-vertex graph as defined in the commentaries, and let $(X, \mathcal{E}')$ be the derived linear graph. Express the number of vertices $v(\mathcal{E}')$ and edges $e(\mathcal{E}')$ in terms of the corresponding numbers for $\mathcal{E}$, and using this show that the Euler characteristic of X is equal to $v(\mathcal{E}) - e(\mathcal{E})$.

2.* Let $(T, \mathcal{E})$ be a locally tame tree in $\mathbf{R}^2$. Using the arguments in the commentaries for Sections 61 and 63, show that $S^2 - T$ is homeomorphic to $\mathbf{R}^2$.

3. If $(X, \mathcal{E})$ is a locally tame graph in $\mathbf{R}^2$, show that $S^2 - X$ has only finitely many components.

4. Let $\mathbf{M}(a, b; c, d)$ be the graph described at the end of the commentary for Section 84. Prove that the fundamental group of $\mathbf{M}(a, b; c, d)$ is a free group on $(b - a) \cdot (d - c)$ generators. [*Hint:* Count the numbers of vertices and edges.]

Munkres, Section 85

Munkres, p. 515: 2
Hatcher, pp. 86–87: 8

Additional exercise

1. Let $E \subset \mathbf{R}^3$ be the set of all points (t, x, y) such that either $x = y = 0$ or else t is an integer and $x^2 + y^2 = 1$, and let $q : E \to S^1 \vee S^1$ be the map sending (t, x, y) to $(\exp(2\pi\mathbf{i}t), x + \mathbf{i}y)$.

(i) Prove that q is an infinite sheeted covering space projection onto the wedge of two circles.

(ii) Prove that the fundamental group of E is a free group on infinitely many generators. [*Hint:* Let F_n denote the portion of E for which $|t| \le n$. Find the fundamental group of F_n, explain why the homomorphism induced by inclusion sends the fundamental group of F_n injectively to the fundamental group of F_{n+1}, and imitate the argument in the commentary for this section of the text.]