HINTS, CORRECTIONS AND SOLUTIONS TO

SELECTED EXERCISES FOR

MATHEMATICS 205B

Winter 2008

PART 1

An "**M**" in front of a number indicates an exercise from Munkres, and an "**A**" in front of a number indicates an exercise from `math205Bexercises.pdf`.

Munkres, Section 52

M6. *Further hints.* (i) The binary operation on the set of closed curves is defined by pointwise multiplication. Therefore, checking the identities $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ and $1 \otimes f = f = f \otimes 1$ reduce to checking that the functions obtained by evaluating both sides at an arbitrary element of G are the same. Similarly, if we define f^{-1} by $f^{-1}(x) = \big(f(x)\big)^{-1}$, then checking that $f \otimes f^{-1} = 1 = f^{-1} \otimes f$ reduces to evaluating all the relevant expressions at an arbitrary point x.

(ii) We need to show that if $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $f_0 \otimes g_0 \simeq f_1 \otimes g_1$, where $\simeq$ denotes endpoint preserving homotopy. Let H and K be the homotopies for the maps f_i and g_i respectively. Consider the homotopy $L = H \cdot K$ defined by the algebraic product of H and K viewed as maps into the topological group G.

(iii) Let $\simeq$ be as above, and let C denote the constant closed curve sending all points to 1. Given $a, b \in \pi_1(G, 1)$, choose representatives α and β respectively. Verify each of the relationships in the chain $\alpha + \beta \simeq (\alpha + C) \otimes (C + \beta) \simeq \alpha \otimes \beta$.

(iv) In the notation of the previous paragraph, verify each of the relationships in the chain $\alpha \otimes \beta \simeq (\alpha + C) \otimes (C + \beta) \simeq (C + \beta) \otimes (\alpha + C) \simeq \beta \otimes \alpha$. Recall that $1 \cdot g = g = g \cdot 1$ for all $g \in G$.

A1. *Correction.* In the second sentence, "$[P, Y]$" should be replaced by "$[P, X]$."

A4. *Further hints.* (i) For each point $x \in U$ there is a $\delta(x) > 0$ such that the open disk $N_{\delta(x)}(x; \mathbf{R}^n)$ is contained in U; it follows that the closures of the disks V_x $N_{\delta(x)/2}(x; \mathbf{R}^n)$ are compact and contained in U. The disks V_x form an open covering of U, and since U is second countable it has the Lindelöf property: Every open covering has a countable subcovering. Let $\{ V_m \}$ denote a countable subcovering extracted from the V_x's, and let F_m be the closure of V_m (hence F_m is a closed disk contained in U). Let

$$C_m \quad = \quad \bigcup_{j=1}^{m} F_m$$

and verify this family of compact subsets has all the required properties.

(ii) Choose $f : X \to Y$ representing u. Why is the image of f contained in some subset C_m?

(iii) Suppose that f and g are homotopic maps from X into C_m and H is the homotopy between $j_m \circ f$ and $j_m \circ g$. Why is the image of H contained in some subset C_n, and why can we choose n to be greater than or equal to m? Recall that if X is compact then so is $X \times [0,1]$.

Munkres, Section 53

M2. *Further hints.* How are the slices of the covering map p over the open set U related to the connected components of $p^{-1}[U]$?

M4. *Further hints.* Let $z \in Z$, write $p^{-1}[\{z\}] = \{z_1, \cdots, z_m\}$, and let U be an open neighborhood of z in Z which is evenly covered with respect to p. Then $p^{-1}[U]$ is a union of pairwise disjoint open subsets U_i ($1 \leq i \leq m$) such that $z_i \in U_i$. Why are there open subsets $V_i \subset U_i$ such that $z_i \in V_i$ and V_i is evenly covered with respect to q? Let

$$W = \bigcap_{i=1}^{m} p[V_i]$$

and explain why W is an open neighborhood of z that is evenly covered with respect to the composite.

M6. *Further hints.* (i) There are several separate conclusions depending upon the assumption on the topology of the codomain B, and it is best to handle each one individually. Another fact along these lines worth noting is that if B is $\mathbf{T_1}$, then so is E (try proving this — the argument is easier than any of the following).

Suppose that B is HAUSDORFF. It is convenient to split things into two cases, depending upon whether or not the distinct points $x, y \in E$ map to the same point in B. If they do, try to construct disjoint neighborhoods using slices of an evenly covered neighborhood of $p(x) = p(y)$. On the other hand, if $p(x) \neq p(y)$, let U and V be disjoint open neighborhoods of these image points in B and consider $p^{-1}[U]$ and $p^{-1}[V]$.

Suppose that B is REGULAR. Let $x \in E$, and let U be an open neighborhood of x in E. Let W be an open neighborhood of $p(x)$ in B that is evenly covered, let W_0 be the unique slice in $p^{-1}[W]$ containing x, and let $W_1 = U \cap W_0$, so that p maps W_1 homeomorphically to an open neighborhood of $p(x)$ which is contained in W. Since B is regular, there is an open neighborhood V_0 of $p(x)$ in B such that

$$p(x) \in V_0 \subset \overline{V_0} \subset p[W_1] .$$

Let $V = W_1 \cap p^{-1}[V_0]$. By construction we have $x \in V$, so it is only necessary to show that the closure of V in E is contained in W_1. — If we let $A = W_1 \cap p^{-1}\left[\overline{V_0}\right]$, then it will suffice to prove that A is closed in E or equivalently that $E - A$ is open in E. Let W' be the union of all the slices for p over $p[W_1]$, **except** for W_1 itself. Check that $E - A$ is the union of W' and $E - \left[\overline{V_0}\right]$, and explain how this is relevant.

Suppose that B is COMPLETELY REGULAR. Let $x \in E$, and let U be an open neighborhood of x in E. Let W be an open neighborhood of $p(x)$ in B that is evenly covered, let W_0 be the unique slice in $p^{-1}[W]$ containing x, and let $W_1 = U \cap W_0$, so that p maps W_1 homeomorphically to an open neighborhood of $p(x)$ which is contained in W. Since B is completely regular, by the preceding discussion we know that E is also regular, so choose an open neighborhood V of x such that the closure of V is contained in W_1. Since B is completely regular and p maps W_1 homeomorphically to a subspace of B, we know that W_1 is also completely regular (recall that a subspace of a completely regular space is also completely regular). Let f_0 be a continuous function from W_1 to $[0,1]$ which is 1 at x and zero on $W_1 - V$, and extend it to E by setting $f = 0$ off $E - V$. Why does this yield a well-defined continuous mapping on E with the required properties? Recall that $W_1 \subset U$ so that $E - U \subset E - W_1$.

Suppose that B is LOCALLY COMPACT HAUSDORFF. Let $x \in E$, and let U be an open neighborhood of x in E. Let W be an open neighborhood of $p(x)$ in B that is evenly covered, let W_0 be the unique slice in $p^{-1}[W]$ containing x, and let $W_1 = U \cap W_0$, so that p maps W_1 homeomorphically to an open neighborhood of $p(x)$ which is contained in W. Since B is locally compact and Hausdorff, there is an open neighborhood V_0 of $p(x)$ such that the closure of V_0 is compact and contained in $p[W_1]$. Explain why $p^{-1}[V_0] \cap W_1$ has a compact closure and is contained in W_1, and also explain how one can finish the proof knowing this.

(ii) This problem is much easier if we are allowed to assume that B is Hausdorff, so we shall do this case first and then do the general case.

THE HAUSDORFF CASE. Let $b \in B$ and let U_b be an evenly covered open neighborhood of b. Then there is an open neighborhood $V_b \subset U_b$ such that the closure of V_b in B is compact and contained in U_b. Take a finite subcovering of B by sets U_i and let V_i be the corresponding smaller open subsets with compact closures. Show that each $p^{-1}[\overline{V_i}]$ is a finite union of compact subsets homeomorphic to $\overline{V_i}$. Why does this imply that E is a finite union of compact sets, and why is a finite union of compact sets also compact?

THE GENERAL CASE. Let $\mathcal{U}$ be a finite open covering of E, let $b \in B$, and let W_b be an evenly covered open neighborhood of b in B. For each $e \in E$ let N_e be an open neighborhood of e such that N_e is contained in a slice over $W_{p(e)}$ and N_e is also contained in some open subset U_α belonging to $\mathcal{U}$. Let W'_b be the open neighborhood of b given by

$$\bigcap_{p(e)=b} p[N_e]$$

and let V_e denote the intersection of N_e and $p^{-1}[W'_{p(e)}]$. Then the sets W'_b form an open covering of B and hence there is a finite subcovering by some sets $W_{b(k)}$ for $1 \leq k \leq m$. Let $\mathcal{V}$ be the family of all open sets V_e where e runs through all points such that $p(e) = b(k)$ for some k. Why is this set finite? Explain why

$$p^{-1}\left[W_{b(k)}\right] \quad = \quad \bigcup_{p(e)=b(k)} V_e$$

and use this to show that $\mathcal{V}$ is a finite open covering of E. For each V_e in $\mathcal{V}$ choose the U_α in $\mathcal{U}$ such that $V_e \subset N_e \subset U_\alpha$, and explain why this collection defines a finite subcovering of E.

A1. *Correction.* There should be an additional hypothesis that E_2 is **connected**. There are simple counterexamples if this condition is not met. Specifically, let $E \amalg E$ denote the (topological) disjoint union of two copies of E, let $q : E \to B$ be a covering space projection, let $p_2 : E \amalg E \to B$ be the map whose restriction to each copy of E is given by q, and let $p : E \to E \amalg E$ be the map which sends E to the first disjoint copy of E on the right hand side. Then $p_1 = p$ and p_2 are both covering space projections, but p is not because it is not surjective.

Hints. First of all, we need the following basic fact:

LEMMA. *Let X be a connected space, and let $\mathcal{B}$ be a base for the topology on X. Then for each $u, v \in X$ there is a finite sequence of open sets $U_0, \cdots, U_m$ in $\mathcal{B}$ such that $u \in U_0$, $v \in U_m$ and $U_i \cap U_{i-1} \neq \emptyset$ for all $i > 0$.*

Sketch of proof. Given an arbitrary X and $\mathcal{B}$, define a binary relation $\mathcal{R}$ by $u \,\mathcal{R}\, v$ if and only if the condition in the Lemma holds, show that it is an equivalence relation, explain why the equivalence classes are open, using this explain why they must also be closed, and finally conclude that there is only one equivalence class if X is connected.■

Returning to the original problem, let $x \in E_2$, let U_1 be an evenly covered open neighborhood of $p_2(x)$ in X with respect to p_2, let U_2 be an evenly covered open neighborhood of $p_2(x)$ with respect to p_1, and let $U = U_1 \cap U_2$, so that U is evenly covered with respect to both maps. Therefore we know that $p_1^{-1}[U]$ and $p_2^{-1}[U]$ are both isomorphic to disjoint unions of copies of U. More precisely, we know that $p_1^{-1}[U]$ is homeomorphic to $U \times A$ for some discrete space A, while $p_2^{-1}[U]$ is homeomorphic to $U \times B$ for some discrete space B, and under these homeomorphisms the map p sends a slice $U \times \{a\}$ to a slice $U \times \{h(a)\}$ by the standard mapsending (u, a) to $\big(u, h(a)\big)$, where $h : A \to B$ is some mapping of indexing sets. If x lies in the slice corresponding to $U \times \{b\}$, then x will be evenly covered, with slices given by all $U \times \{a\}$ such that $h(a) = b$. It follows that p will be a covering map provided it is surjective. The point of the connectedness condition is that it should imply the surjectivity of p.

To show this, proceed as follows: Suppose that we have $z \in E_2$ and z lies in an open set U such that $p_2[U]$ is evenly covered with respect to both p_i. Use the discussion of the preceding paragraph to show that if z lies in the image of p_1 then so does every point in U. Let $\mathcal{B}$ be the base of open sets in E_2 satisfying the condition in the second sentence of this paragraph, let $\mathcal{R}$ be as in the lemma, and show that if two points lie in the same equivalence class with respect to $\mathcal{R}$ and one of them lies in the image of p, then so does the other. Why does this imply that p is onto? Recall that E_2 is connected.

A2. *Hints.* (i) Given $y \in Y$, let $V \subset X$ be an evenly covered open neighborhood of $f(y)$, and let U be an open neighborhood of y such that $f[U] \subset V$.

(ii) Let h be the continuous mapping sending (y, e) to e; explain why the image of h lies in the inverse image of Y, verify the functional identities in the exercise, and show that an inverse to h is given by $k : p^{-1}[Y] \to Y \times_X E$ is given by $k(z) = (\, p(z), e\,)$. One needs to check that the formula determines an element in the subspace $Y \times_X E \subset Y \times E$.

A3. *Hints.* By the assumptions, the topology for X has a base of open subsets that are also closed; explain why it has a base of evenly covered open subsets that are also closed. Consider the slices in E which lie above such open subsets of X. We claim they form a base for the topology on E; explain why it suffices to show that each slice is closed in E. To show that such a slice V is closed, write its complement $E - V$ as a union of the sets $E - p^{-1}[\,p[V]\,]$ and all the other slices V' such that $p[V'] = p[V]$? Why are all these subsets open in E?

Munkres, Section 54

M8. *Hints.* Before proceeding, it is useful to note the following general result:

PROPOSITION. *If $p : E \to B$ is a covering space projection, then p is an open mapping.*

Proof. Let $U \subset E$ be open, for each $y \in U$ let V_y be an open neighborhood of $f(y)$ which is evenly covered, and let U_y be an open neighborhood of y such that $U_y \subset U$ and U_y is contained in a slice over V_y. Then by definition the map p sends U_y to an open subset of B, and hence

$$p[U] \quad = \quad \bigcup_{y \in U} p[U_y]$$

is open in X.■

Returning to the original problem, we know that p is open, continuous and onto, so it is only necessary to check that p is 1–1. The general results on covering spaces imply that for each $b \in B$ the inverse image $p^{-1}[\{b\}]$ is in 1–1 correspondence with the set of cosets $\pi_1(B, b)/\text{Image } p_*$. Why do this and simple connectivity imply about $p^{-1}[\{b\}]$?

A1. *Hints.* Let $p_n : \mathbf{R}^n \to T^n$ be the Cartesian product of n copies of $p : \mathbf{R} \to S^1$. Then p_n is equivalent to $p_{n-1} \times p$ under the natural idenfications of $\prod^n X$ with $\left(\prod^{n-1} X\right) \times X$ when specialized to $X = \mathbf{R}$ or S^1. We know that $p = p_1$ is a covering space projection, and by a result from Section 53 we know that p_n will be a covering space projection if p_{n-1} is.

A2. *Further hints.* We know that $\pi_1(T^n, e) \cong \mathbf{Z}^n$. Why are the automorphisms of this group equal to the set of matrices described in the problem? To work the second part, take the map described in the original hint. The associated map will then be given by an $n \times n$ matrix over the integers. To determine the (i, j) entry of this matrix, let $\theta_j : S^1 \to T^n$ denote the injection map whose projection onto the j^{th} coordinate is the identity and whose projection onto the other coordinates is the constant map with value 1, and let p_i denote projection onto the i^{th} coordinate. Find a relationship between $p_i \circ f_A \circ \theta_j$ and the entries of A.