HINTS, CORRECTIONS AND SOLUTIONS TO

SELECTED EXERCISES FOR

MATHEMATICS 205B

Winter 2008

PART 2

The same conventions apply as in Part 1, and furthermore an "**H**" in front of a number indicates an exercise from Hatcher which is listed for the appropriate section in `math205Bexercises.pdf`.

Munkres, Section 55

M1. *Hint.* Let $i : A \to D^2$ be the retract map, let $r : D^2 \to A$ be the retraction, and consider $i \circ f \circ r$.

M2. *Hint.* Why does h extend to a continuous map from D^2 to S^1, and what can one conclude about the maps $i \circ h$ and $-i \circ h$, where i is the inclusion?

M3. *Hint.* Explain why f maps the closed first octant in $\mathbf{R}^3$ to itself, and define a related continuous self-map on the intersection of this set with the unit sphere in $\mathbf{R}^3$ as in the proof of Corollary 55.7.

H10. *Hint:* Consider the special case where f is the identity map. If every map to or from X is nullhomotopic, then the identity map is nullhomotopic. Conversely, explain why every map to or from X is nullhomotopic if the identity map is.

H13. *Hint:* Suppose that a_1 and a_2 are in A, and choose a curve γ_1 in A joining them (this exists since A is arcwise connected). Now let γ_2 be an arbitrary curve in X joining a_1 and a_2, so that $(-\gamma 1) + \gamma_2$ is a closed curve based at a_2 If the map i_* from $\pi_1(A)$ to $\pi_1(X)$ is onto, then this is base point preservingly homotopic to a closed curve β in A. Explain why $\gamma_1 + \beta$, whose image lies in A, is end point preserivingly homotopic in X to γ_2.

H17. *Hints.* Take the initial retract from S^1 to T^2 to be the slice inclusion $j(z) = (z, 1)$. Why are the maps $r_n(z, w) = z \cdot w^n$, where n runs through the integers, all retractions for j, and how can one use the associated maps of fundamental groups to show that the maps r_n are not homotopic?

Munkres, Section 58

M2. *Comment.* As stated the problem does not require full proofs, and in any case the most important thing is to describe approaches to showing that the fundamental groups are isomorphic to a specific choice among the three alternatives.

M7. *Hints.* PART (a) : Why is the map of homotopy classes $j_* : [X, A] \to [X, X]$ injective, why is $j_* \circ f_*$ the identity on $[X, X]$, and why does this imply that j_* is also surjective?

PART (b) : Same questions as in (a), but different reasons are needed.

PART (c) : Look for an example where X is contractible but A is not, so that the map $j \circ f$ is automatically homotopic to the identity.

M9. *Hint.* Let $t_0 \in \mathbf{R}$ be such that $p(t_0) = x_0$, and consider the unique lifting β of $h \circ \gamma$ to a curve $[0,1] \to \mathbf{R}$ such that $\beta(0) = t_0$. Why is $\beta(1) = t_0 + n$ for some integer n, and why is n equal to the degree as defined in the exercise? One can use this to dispose of all parts of the exercise except (c). To prove the latter, use (b) to show that it suffices to consider cases in which h and k are given by raising z to some integral power, and use (d) plus the standard laws of exponenets to finish the argument.

H1. *Hint.* Explain why the torus is homeomorphic to the quotient of $K = [0,1] \times [0,1]$ modulo the equivalence relation defined by setting $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$, construct a retraction from $K - \{(\frac{1}{2}, \frac{1}{2})\}$ to the boundary of K, and explain why this passes to a continuous retraction of quotient spaces.

H4. *Hint.* Let $r : X \to A$ be the map defined by f_1, and show that $r|A$ is homotopic to id_A.

H5. *Hint.* Let $H : X \times [0,1] \to X$ be the homotopy from the identity to the constant map; Hatcher's definition of a deformation retract means that the restriction of H to $\{x\} \times [0,1]$ is constant. Why is there an open neighborhood V of x such that $V \subset U$ and H maps $V \times [0,1]$ into U? Remember that $\{x\}$ is compact.

H9. *Hint.* Look at the composite of the contracting homotopy and the retraction.

H13. *Change of notation.* Let $i : A \to X$ denote inclusion, let ρ and σ denote the two retractions from $X \to A$, and let H and K denote the homotopies from $i \circ \rho$ and $i \circ \sigma$ to the identity.

Hint. Why is the map of homotopy classes $i_* : [X, A] \to [X, X]$ injective, and what does this imply about ρ and σ?

A4. *Background material.* For many purposes in topology it is useful to formalize some basic facts about connected components in a setting similar to that for fundamental groups of spaces. Given a space X, let $\mathbf{CC}(X)$ denote the set of connected components of X. Since the continuous image of a connected set is connected, it follows that if $f : X \to Y$ is a continuous map then there is a well-defined map of sets $f_* : \mathbf{CC}(X) \to \mathbf{CC}(Y)$ such that if A is the connected component of $x \in X$, then $f_*(A)$ is the connected component of Y which contains $f(x)$ and hence also $f[A]$. It follows immediately that if f is the identity map on a space X, then f_* is the identity on the set of connected components, and also if $g : Y \to Z$ is another continuous map then $(g \circ f)_* = g_* \circ f_*$. Furthermore, if f and g are homotopic maps from one space X to a second space Y, then $f_* = g_*$.

Similarly, one can define the set of arc components $\mathbf{AC}(X)$ for a space X and an associated map of sets $f_* : \mathbf{AC}(X) \to \mathbf{AC}(Y)$, and these maps will have all the properties described in the previous paragraph. For each space X there is also a well-defined *natural transformation* $\theta_X : \mathbf{AC}(X)$ such that for every continuous mapping $f : X \to Y$ we have $f_* \circ \theta_X = \theta_Y \circ f_*$. Verification of these statements is left to the reader as an exercise.

Given a point x in a space X and a nonnegative integer n, we shall say that $\mathbf{L}(X; x) = n$ if the following holds:

> If U is an open neighborhood of x, then there is an open neighborhood $V \subset X$ such that $\mathbf{CC}(V - \{x\}) = n$ and for all open neighborhoods W such that $x \in W \subset V$ the map from $\mathbf{CC}(W - \{x\})$ to $\mathbf{CC}(V - \{x\})$, induced by the inclusion of $W - \{x\}$ in $V - \{x\}$, is onto.

EXAMPLES.

1. If U is open in $\mathbf{R}^n$ and $n \geq 2$, then for each $x \in U$ we have $\mathbf{L}(X;x) = 1$.

2. If J is an interval in the real line, then we have $\mathbf{L}(J;t) = 2$ for all $t \in J$ that are not endpoints and $\mathbf{L}(J;t) = t$ if $t \in J$ is an endpoint.

3. If $f : X \to Y$ is a homeomorphism and $x \in X$ is such that $f(x) = y$, then $\mathbf{L}(X;x) = n$ if and only if $\mathbf{L}(Y;y) = n$.

In particular, the second and third statements combine to give a proof that if J_1 and J_2 are intervals in the real line and $f : J_1 \to J_2$ is a homeomorphism, then f must send endpoints to endpoints and non-endpoints to non-endpoints. This is a slightly stronger version of the standard assertion that the number of endpoints on an interval is topologically invariant. Likewise, all three statements give a more formal way of stating the standard proofs that an interval in the real line is not homeomorphic to an open subset in $\mathbf{R}^n$ for $n \geq 2$.

The preceding definition is particularly useful for studying a class of spaces called *finite edge-vertex graphs*. These are Hausdorff spaces Γ which are finite unions of subsets E_i such that each E_i is homeomorphic to $[0,1]$ and if $i \neq j$ then $E_i \cap E_j$ is finite and consists of endpoints for both subsets (say, as given by the preceding discussion — it follows that the intersection is either one point which is an endpoint of each or else it two points and is the set of endpoints for each of the two intervals). It follows that a if a point $x \in E_i$ also lies in E_j then it is also an endpoint of E_j, so that it is meaningful to talk about the set V of points that are endpoints for the intervals in the collection $\mathcal{E} = \{E_i\}$; the elements of $\mathcal{E}$ are called the *edges* of Γ with respect to $\mathcal{E}$, and the elements of $\mathbf{V}$ are called the *vertices* for Γ with respect to $\mathcal{E}$. Clearly the set of vertices is finite. The next two properties follow directly from the definitions:

(A) *If X is a finite edge-vertex graph, then X is locally arcwise connected and has finitely many (arc) components.* (This follows immediately for non-endpoints, which all have open neighborhoods homeomorphic to open intervals. For vertices, observe that each vertex v has an open neighborhood of the form $\cup_j\ B_j$, where each B_j is a closed subset homeomorphic to a half open interval, the vertex v corresponds to the unique endpoint of each B_j, and the subsets $B_j - \{v\}$ are pairwise disjoint.

(B) *For each $x \in \Gamma$ we have $\mathbf{L}(\Gamma;x) = 2$ if x is not a vertex, and for each vertex $v \in \mathbf{V}$ we have $\mathbf{L}(\Gamma;v) = n$, where n is the number of edges E_i that contain v.*

Using the statement in Example 3 above, we can often decide very easily whether or not two edge-vertex graphs are homeomorphic. For example, consider the unit circle defined by the equation $x^2 + y^2 = 1$, which is formed from the upper and lower semicircles $E_\pm$ of points where the second coordinate is nonnegative or nonpositive. For this space we have $\mathbf{L}(\Gamma;x) = 2$ for all x. On the other hand, for the closed interval $[0,1]$ we know that $\mathbf{L}(\Gamma;x) = 2$ for all x such that $0 < x < 1$, while $\mathbf{L}(\Gamma;x) = 1$ for $x = 0, 1$. One efficient way of proceeding is as follows: We know that for all but at most finitely many $x \in \Gamma$ we have $\mathbf{L}(\Gamma;x) = 2$. Thus for all positive integers $n \neq 2$ we know that the number $N(\Gamma, n)$ of vertices v with $\mathbf{L}(\Gamma;v) = n$ is a positive integer. For homeomorphic (edge-vertex) graphs, these sequences must be the same, so one way of showing that two graphs Γ_1 and Γ_2 are not homeomorphic is to show that the sequences $N(\Gamma_1, n)$ and $N(\Gamma_2, n)$ have different values for some choice of n. In particular, if we agree that $N(\Gamma, 2)$ is always ∞, then for a graph homeomorphic to a Figure 8 we have $N(n) = 0, \infty, 0, 1, 0,\ \cdots$ while for a graph homeomorpic to a Figure Theta we have $N(n) = 0, \infty, 2, 0,\ \cdots$, which shows that the two spaces cannot be homeomorphic.

Hint for the problem: Compute the numbers $N(n)$ for each of the spaces described in the problem, and see how many different sequences are realized.

Munkres, Section 59

M1. *Hint.* Thicken the two spheres into open subsets U and V such that the first sphere is a deformation retract of U, the second is a deformation retract of V, and the intersection is arcwise connected.

M2. *Hint.* Recall that there is a continuous curve in the plane whose image is D^2.

M3. *Hint.* If $f : X \to Y$ is a homeomorphism such that $f(x) = y$, then $X - \{x\}$ and $Y - \{y\}$ are homeomorphic.

Munkres, Section 60

M5. *Hint.* If the fundamental group of the Figure 8 space is abelian, why is the fundamental group of every connected covering space also abelian, and why are all these fundamental groups isomorphic to subgroups of $\mathbf{Z} \times \mathbf{Z}$? Let E be the union of the four circles A_i and B_i, explain why $B_1 \cup A_0$ and $A_1 \cup B_0$ are (pointed) retracts of E, and show that the retractions can be used to define a surjection from $\pi_1(E)$ to the group $\mathbf{Z}^4$. Why would this yield a contradiction if the fundamental group of the Figure 8 space was abelian?

H18(a)**.** *Preliminary remark.* We have not yet discussed attaching cells in the course, but this part of the problem can be worked using only the material in Section 59 of Munkres.

Hints. Thicken the copies of S^1 and S^2 into open subsets U and V such that S^1 is a deformation retract of U, S^2 is a deformation retract of V, and $U \cap V$ is arcwise connected. Use this to prove that the fundamental group of $S^1 \vee S^2$ is cyclic. Show that S^1 is a retract of the one point union, and using this explain why the fundamental group must be infinite.

H2. *Hints.* Give a proof by induction on $n \geq 2$, modifying the hypothesis to include an assumption that every pair of open subsets has a nonempty intersection. Why is the result true when $n = 2$?

H8. *Comment.* It was premature to assign this problem, so disregard it for the time being.