HINTS, CORRECTIONS AND SOLUTIONS TO

SELECTED EXERCISES FOR

MATHEMATICS 205B

Winter 2008

PART 3

Munkres, Section 61

M2. *Hints.* The basic constructions for our proof are described in the file `polishcircle.pdf`. Our solution to the exercise will also use the following basic consequence of the proof of the Jordan Curve Theorem:

> *Let C be a locally flat simple closed curve in $\mathbf{R}^2$, let $x \in C$, and let U be an open neighborhood such that $U - C$ has exactly two components. If y and z lie in different components of $U - C$, then they also lie in different components of $S^2 - C$.*

Let Φ be the model for the Polish circle described in `polishcircle.pdf`. Then if W is the open neighborhood of $(0, -2) \in \Phi$ given by

$$\left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{5}{2}, -\frac{3}{2}\right)$$

then $W - \Phi$ has two components, one of which contains the point $\mathbf{y} = \left(\frac{1}{4}, -\frac{7}{4}\right)$ and the other of which contains the point $\mathbf{z} = \left(-\frac{1}{4}, -\frac{9}{4}\right)$.

We shall use the notation of `polishcircle.pdf` freely, and particular, we shall need the subspaces of the plane C_n and B_n constructed there. Since C_n is a simple closed piecewise smooth curve, it is locally flat and hence our version of the Jordan Curve Theorem applies (we are assuming Exercise A2 below). Suppose now that γ is a closed curve joining $\mathbf{y}$ to $\mathbf{z}$; we need to show that the image A of γ contains a point of P.

Using the Jordan Curve Theorem and the observation stated above, show that A must contain at least one point of C_n for every n. Why does this imply that A also contains at least one point of B_n for each n? Therefore all sets in the descreasing family of compact sets $B_n \cap P$ are nonempty. Use compactness to show that their intersection is nonempty, and explain why this intersection is $A \cap P$.

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M1. *Comment.* We shall assume that both simple closed curves are locally flat.

Hints. View S^2 as the one point compactification of $\mathbf{R}^2$. We might as well assume that neither curve contains the point at infinity. The curve C_1 lies either in the bounded component of $S^2 - C_2$ or in its unbounded component. In the second case, let c be a point in the bounded component and consider the map h from S^2 to itself which switches c and ∞, and on other points it is defined by $1/(z-c)$. Why does this define a homeomorphism from the sphere to itself, and why does $h[C_1]$ lie in the bounded component of $h[C_2]$? Using this, explain why it suffices to consider cases where the first curve lies in the bouned component of the second curve's complement. — Let U_i and V_i be the bounded and unbounded components of $S^2 - C_i$, where $i = 1, 2$. Explain why U_1 and V_1 contain points of U_2. Why is C_1 disjoint from V_2, and how can one use this to show that $V_2 \subset V_1$? Why does every point of C_2 contain a neighborhood which is disjoint from $U_1 \cup C_1$? Combine these to show that S^2 is the union of the pairwise disjoint open sets U_1, $V_1 \cap U_2$ and V_2. Show that the second part of the exercise follows immediately. The only remaining point is to prove that $V_1 \cap U_2$ is connected. Explain why this set has connected components W_1 and W_2 such that $\Omega_1 = W_1 \cup C_1 \cup U_1$ and $\Omega_2 = W_2 \cup C_2 \cup V_2$ are open in S^2. Explain why $U_1 \cup \Omega_1 \cup \Omega_2 \cup V_2$ is an open and closed subset of S^2 and hence is all of S^2. Finally show that $\Omega_1 \neq \Omega_2$ implies that $U_1 \cup \Omega_1$ and $V_2 \cup \Omega_2$ separate S^2. — This argument has a great deal in common with the proof of Lemma 64.1 given in the commentaries.

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M1(a). *Hint:* More generally, one can show that if we are given finitely many closed subsets $F_i \subset X$ whose union is X, then X is Hausdorff if and only if each F_i is.