HINTS, CORRECTIONS AND SOLUTIONS TO

SELECTED EXERCISES FOR

MATHEMATICS 205B

Winter 2008

PART 4

Munkres, Section 68

M2(b)**.** *Hint.* A reduced word of odd length ≥ 3 has the form $x_1 y_1 x_2 \ \cdots \ y_n x_{n+1}$ where $x_i \in G_1$ and $y_j \in G_2$ or vice versa.

Munkres, Section 69

M1. *Hints.* If we define an *abelianization* of G to be a homomorphism $\alpha : G \to A$ such that (a) A is abelian, (b) for every $\beta : G \to B$, where B is abelian, there is a unique homomorphism $h : A \to B$ such that $h \circ \alpha = \beta$, prove that the following hold:

(1) The quotient projection $G \to [G,G]$ is an abelianization.
(2) If α and α' are abelianizations, then there is a unique isomorphism $\varphi : A \to A'$ such that $\varphi \circ \alpha = \alpha'$.

Complete the proof by showing that the direct sum group is an abelianization of $G_1 \times G_2$.

M3. *Hints.* Explain why we may, without loss of generality, assume that $m \geq n$.

Munkres, Section 70

M1. *Hint:* (a) In general, if we are given group homomorphisms $f_1 : K \to G_1$ and $f_2 : K \to G_2$, the pushout group P is the quotient of $G_1 * G_2$ modulo the normal subgroup N generated by all elements of the form $f_1(a)\, f_2(a)^{-1}$ for $a \in K$. Let $q_i : G_i \to P$ be the composites of inclusions into the free product followed by projections onto the pushout. The triviality of the composite $q_2 \circ f_2 = q_1 \circ f_1$ implies that each q_i factors through a map r_i from G/N_i to P, and by the Universal Mapping Property of free products these define a unique homomorphism from $G/N_1 * G_2/N_2$ to P. This map is onto because the images of G_1 and G_2 generate P.

(b) Let M be the normal subgroup of $G_1 * G_2$ generated by $N_1 * N_2$. Then M contains the normal subgroup N and hence there is a canonical homomorphism from P to the quotient by the subgroup M. Show that

$$G_1 * G_2/M \quad \cong \quad G/N_1 * G_2/N_2$$

and that the composite homomorphism

$$G/N_1 * G_2/N_2 \ \longrightarrow \ P \ \longrightarrow \ G_1 * G_2/M \ \cong \ G/N_1 * G_2/N_2$$

is the identity.

A1.(i) *Hint:* The abelianization may be written additively as the quotient group of the free abelian group $\mathbf{Z} \oplus \mathbf{Z}$ by the image of the homomorphism from $\mathbf{Z}$ which sends a generator to $(3, 2)$. Show that this quotient is infinite cyclic.

Munkres, Section 71

M4. *Hint:* Suppose the space is first countable, and let $\{ U_n \}$ be a countable base at the intersection point of the circles. Why is the intersection of U_n with the n^{th} circle an open subset of that circle? If $\{x_n\}$ is an arbitrary sequence of points in X such that x_n lies in the n^{th} circle for all n, why is the complement of the infinite set $\{x_1, x_2, \cdots \}$ an open subset of X (in other words, why is the given set closed)? Consider what happens if one chooses x_n to be a point in the intersection of U_n with the n^{th} circle that is not the common point.

M5. *Hints:* (a) Show that one of X, Y is metrizable but the other is not. Also show that neither of these spaces is compact but the infinite earring in Example 71.1 is compact.

(b) Let Y_n be the union of the first n circles. Why is the fundamental group of Y_n free on n generators, and why do the images of these free generators in $\pi_1(Y_{n+1})$ extend to a set of free generators for the latter group? Explain why every compact subset of Y is contained in an open subset which is homotopy equivalent to some Y_n (look at the sets on which the first coordinate is less than $2n + \frac{1}{2}$).

H20. *Comment:* One can construct a continuous 1–1 onto map from X to Y which sends the first circle in X to the first circle in Y, and so on; verification of continuity depends upon the definition of the topology in the infinite wedge. Although this map cannot be a homeomorphism (see the preceding hint), it is a homotopy equivalence. — Proving that the map is a homotopy equivalence using the methods of this course seems to be extremely challenging at best, so this part of the problem should be disregarded.

Munkres, Section 72

M1. *Hints.* Recall that S^{n-1} is simply connected if $n > 2$.

Munkres, Section 59

H8. *Hints.* Let U_1 be the open set given by points in the image of the first torus plus points which lie in the image of $S^1 \times V$ in the second, where V is a small arc centered at x_0, and let U_2 be formed similarly with the roles of the two tori interchanged. Explain why the first torus is a strong deformation retract of U_1 and the second is a strong deformation retract of U_2. Why is the common circle a strong deformation retract of the intersection? Apply the Seifert-van Kampen Theorem.