# Review — II

Here are some suggestions for review in connection with the final examination to be held on Thursday, March 20, 2008.

About one sixth of the exam will be material that was also covered in the midterm examination, and the rest will cover "new" material. As before, the main references are the sections in Munkres listed elsewhere, the accompanying material in `commentaries.pdf`, the exercises listed in `math205Bexercises.pdf`, and the other course-related documents in the course directory.

A few of the problems will be straightforward repetitions or examples of a sort already considered, but about half the problems will be new in the sense that they were not explicitly stated previously. In each case, these will be fairly simple extensions of things covered in the course, usually involving an understanding of basic definitions and results and the ability to take things further with two or three additional simple logical steps.

Here are some comments about priorities. For Chapter 10, the important thing is to know the full statement of the Jordan Curve Theorem and the concept of a locally flat curve in the plane. It is good to recognize the meaning of this for curves joining two points in the complement of a simple closed curve in the plane (if they lie in the same component, then one can join them with a curve that misses the original curve, but if not then every curve in the plane joining the points must meet the original curve). For Chapter 11, it is necessary to understand the notions of free products and free groups, the concept of reduced words, the characterizations of free products/groups by universal mapping properties, and of course the statement (but not the proof) of the Seifert-van Kampen Theorem; it is probably also good to understand the latter in terms of the notion of pushout. The most important results in Chapter 13 are the lifting criterion, the classification of (connected) covering spaces over "nice" spaces, and the existence of universal simply connected covering spaces for the latter. For the latter, the existence is the most important thing to recognize, and there will be no questions about the proof itself. The ability to sketch the proof of the lifting criterion in 4 to 6 lines should be viewed as a high priority. Also, the concepts of covering transformations and the result on covering transformations on a universal covering should be understood. Finally, in chapter 14 it is important to understand the notion of a linear graph as defined in Munkres and the notion of a (maximal) tree, as well as the definitions needed to formulate the other. Other important things are the definition of the Euler characteristic for the graph, the relation between the latter and the theorem describing the fundamental group of a connected graph, the result on the Euler characteristic of a tree, and the ability to describe abstractly the fundamental group of a connected graph for specific examples like the utilities network and the complete graph on five vertiex (these will not be on the exam, but it is likely that similar examples will be!).

As before, none of the starred exercises will be on the examination, but in a few cases there might be questions which involve the statements to be proved in such exercises. Additional comments may be posted in response to questions received during the days before the examination.

**Selected problems**

The following problems were considered for inclusion in the final examination but not chosen, usually because they seemed too difficult or lengthy. However, it might be worthwhile to look at them and use them as a test of how well the course material is understood.

**1.** Let $X$ be an arbitrary topological space, and let $X \times S^1 \subset of X \times D^2$ in the usual fashion. Prove that this inclusion map is not a retract.

**2.** A *weak generalized inverse* to a function $f : X \to Y$ is a function $g : Y \to X$ such that $f = f \circ g \circ f$. The name is motivated by various concepts of generalized inverses from matrix algebra, and in set theory one can find weak generalized inverses if $X, Y \neq \emptyset$ by the Axiom of Choice. Let $f : S^1 \to S^1$ be the map $f(z) = z^2$, and let $g$ be a base point preserving weak generalized inverse to $f$. Prove that $g$ cannot be be continuous. [*Hint:* Look at the associated maps of fundamental groups.]

**3.** Let $e$ be an idempotent base point preserving map of a pointed space $(X, x_0)$ to itself, and define subgroups $H$ and $K$ of $\pi_1(X, x_0)$ such that $H$ is the image of $e_*$ and $K$ is the kernel of $e_*$. Prove that every element of the fundamental group is a product of an element of $H$ with an element of $K$ in both orders ($g = hk = k'h'$, but $h$ is not necessarily equal to $h'$ and $k$ is not necessarily equal to $k'$), the subgroup $K$ is normal, and $H \cap K = \{1\}$.

**4.** Suppose that $X$ is a Hausdorff, connected, and locally simply connected space, and assume further that $X$ is the union of the two connected open subsets $U \cup V$ such that $U \cap V$ is also connected. If $X$ is simply connected, prove that the normal subgroups of $\pi_1(U)$ and $\pi_1(V)$ which are normally generated by the images of $\pi_1(U \cap V)$ must be $\pi_1(U)$ and $\pi_1(V)$ respectively. [*Hint:* Suppose, say that the normal subgroup $M$ of $\pi_1(U)$ which is normally generated by the image of $\pi_1(U \cap V)$ is a proper subgroup. Using the universality property for $\pi_1(X)$, construct a homomorphism from the latter to $\pi_1(U)/M$ such that the composite map $\pi_1(U) \to \pi_1(X) \to \pi_1(U)/M$ is the standard projection map, and explain why this homomorphism is nontrivial.]

**5.** Let $f_i : G_i \to H_i$ be group homomorphisms for $i = 1, 2$. Using the universal mapping property of free products, prove that there is a unique homomorphism $f_1 * f_2$ from $G_1 * G_2$ to $H_1 * H_2$ such that its restriction to $G_i$ is given by $f_i$ (followed by inclusion into the free product). Also, show that if we are given homomorphisms $g_i : H_i \to K_i$, then

$$(g_1 \circ f_1) * (g_2 \circ f_2) \quad = \quad (g_1 * g_2) \circ (f_1 * f_2) \ .$$

**6.** Suppose that $U$ and $V$ are connected open subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, so that $U \times V$ can be viewed as a connected open subset of $\mathbf{R}^{n+m}$. Explain why $U \times \mathbf{R}^m \cup \mathbf{R}^n \times V$ is simply connected.

**7.** Let $\Gamma$ be a locally flat simple closed curve in the plane, and let $\Gamma'$ be another locally flat curve defined on an open interval. We shall say that the curves *meet transversely everywhere* if for each intersection point $p$ there is an open neighborhood $U$ of $p$ and a homeomorphism from $U$ to some open neighborhood $V$ of $\mathbf{0}$ such that $\Gamma$ corresponds to the intersection of $V$ with the $x$-axis and $\Gamma'$ corresponds to the intersection of $V$ with the $y$-axis. — Suppose now that $\Gamma$ and $\Gamma'$ meet transversely everywhere and we are given two points $\mathbf{a}$ and $\mathbf{b}$ on $\Gamma'$ which do not lie on $\Gamma$, and also that we can parametrize $\Gamma'$ by some $\alpha$ such that $\alpha(a) = \mathbf{a}$, $\alpha(b) = \mathbf{b}$, and the intersection points of the curve between the parameter values $a < b$ occur at $x_1, \cdots, x_n$. Prove that $\mathbf{a}$ and $\mathbf{b}$ lie in the same component of $\mathbf{R}^2 - \Gamma$ if $n$ is even and they lie in opposite components if $n$ is odd.