Exercise on orbit spaces — II

Since the reference for completing the solution to Exercise 31.8 on page 199 of Munkres is probably not quite as complete as claimed in `orbitspaces.pdf`, we shall explain how to work the remaining parts of the problem if G is finite, and we shall indicate how one can extend the methods to cover cases where the topological group G is compact.

According to the exercise, if the finite group G (with the discrete topology) acts on a topological space X and X is one of the following, then so is X/G:

(0) Hausdorff.

(1) Regular.

(2) Normal.

(3) Locally compact.

(4) Second countable.

We already did (0), but it is important to look back at the basic construction appearing in the solution to this case. Given distinct points $u = p(x)$ and $v = p(y)$, we constructed G-invariant disjoint open neighborhoods of $G \cdot \{x\}$ and $G \cdot \{y\}$ in X , and these projected down to disjoint invariant open neighborhoods of u and v in X/G. The same idea works for both (1) and (2).

Suppose that the space X is regular. Let $z \in X/G$, and let $F \subset X/G$ be a closed set which does not contain z. Write $z = p(x)$ and let $E \subset X$ be the closed set $p^{-1}[F]$. By construction we know that E is G-invariant. Since X is regular, for each $g \in G$ there are disjoint open neighborhoods U_g of $g \cdot x$ and V_g of E. Let $V_0 = \cap_g V_g$, and let $U_0 = \cap_g g^{-1} \cdot U_g$. Then $G \cdot U_0$ and $\cap_g g \cdot V_0$ are disjoint G-invariant open neighborhoods of $G \cdot \{x\}$ and E because

$$G \cdot U_0 \ \cap \ V_0 \ = \ \bigcup_{g \in G} (g \cdot U_0) \ \cap \ V_0 \ \subset \ \bigcup_{g \in G} U_g \ \cap \ V_g$$

and each summand on the right hand side is empty. One can now argue as in the Hausdorff case that the images of $G \cdot U_0$ and $\cap_g g \cdot V_0$ under p are disjoint open neighborhoods of z and F.■

Suppose that the space X is normal. Suppose that F_1 and F_2 are disjoint closed subsets of X/G, and let $E_i = p^{-1}[F_i]$. Then the sets E_i are closed, disjoint and G-invariant. Let U_1 and U_2 be disjoint open neighborhoods of E_1 and E_2 respectively, and consider the G-invariant open neighborhoods $V_i = \cap_g g \cdot U_i$. Then we have $V_1 \cap V_2 \subset U_1 \cap U_2 = \emptyset$, and as before the images of V_1 and V_2 under p will be disjoint open neighborhoods of F_1 and F_2.■

Suppose that the space X is locally compact. There is some ambiguity about the definition of local compactness (since we do not necessarily assume that X is Hausdorff), so we shall assume that each point in X has a neighborhood base of open subsets with compact closures. Let $x \in X$, and let U be an open neighborhood of $p(x)$ in X/G. Then $U_0 = p^{-1}[U]$ is an open neighborhood of $G \cdot \{x\}$ in X, and by local compactness there are open neighborhoods V_g of the points $g \cdot x$ such that

$$g \cdot x \ \in \ V_g \ \subset \ \overline{V_g} \ U_0$$

and each closure $\overline{V_g}$ is compact. As usual let $V_0 = \cap_g \ g^{-1} \cdot V_g$, so that $G \cdot V_0$ is G-invariant; by construction the latter is contained in U_0. Furthermore, the closure of $G \cdot V_0$ is just $G \cdot \overline{V_0}$ (why?), and since

$$G \cdot V_0 \quad \subset \quad \bigcup_{g \in G} V_g \quad \subset \bigcup_{g \in G} \overline{V_g} \quad \subset \quad U_0$$

it follows that the closure of $G \cdot V_0$ is contained in the compact set $\cup_g \overline{V_g}$, and hence this closure is compact.

The preceding implies that $p[V_0] \subset U$ is an open neighborhood of $p(x)$. Furthermore, by continuity we have

$$p\left[\overline{V_0}\right] \quad \subset \quad \overline{p[V_0]} \ .$$

The left hand side is compact and contained in $p[U]$. To see that the left hand side is also closed, note that

$$p\left[\overline{V_0}\right] \quad = \quad p\left[G \cdot \overline{V_0}\right] \quad = \quad X/G \ - \ p\left[X - G \cdot \overline{V_0}\right] \ .$$

Now the expression inside the brackets on the right hand side is open since it is the complement of a closed set, and since p is an open mapping it follows that the set on the right hand side is the complement of an open set and hence is closed. Combining this with the preceding observations, we see that $p\left[\overline{V_0}\right]$ is closed and hence is equal to the closure of $p[V_0]$. Thus the latter is an open neighborhood of $p(x)$ which has a compact closure that is contained in $U = p[U_0]$.■

Suppose that the space X is second countable. Let $\mathcal{A}$ be a countable base for the topology of X, and define a family of open subsets $\mathcal{A}^*$ on X/G by the sets $p[V]$, where $V \in \mathcal{A}$. By construction $\mathcal{A}^*$ is countable, so we need to show it is a base for X/G. But if W is open in X/G, then the open set $p^{-1}[W]$ is a union $\cap_\alpha \ U_\alpha$ where each U_α lies in $\mathcal{A}$. It then follows that

$$W \quad = \quad p\left[p^{-1}[W]\right] \quad = \quad \bigcup_\alpha p[U_\alpha]$$

presents W as a union of sets in the countable family $\mathcal{A}^*$, and hence X/G is second countable.■

Extension to nonfinite compact groups

The final case (4) does not require any finiteness or compactness assumption, so nothing more is needed to show that if X is second countable then so is X/G.

In the other cases, here is the additional input that is needed. To prove (0), we need to know that two disjoint compact subsets of a Hausdorff space have disjoint open neighborhoods, and to prove (1), we need to know that if K and F are disjoint compact and closed subsets of a regular space, then K and F have disjoint neighborhoods. In both cases we also need to know that if G is compact then the action map $\alpha : G \times X \to X$ is closed or something of the same sort. The closed mapping result is discussed in Bredon, and a weaker version (which is adequate for solving the exercise) is mentioned in the hint at the bottom of page 199 in Munkres. At various points in our arguments we created invariant open neighborhoods by taking finite intersections of sets $g \cdot A$ for $A \subset X$. Obviously this will not work if G is infinite, and again methods for dealing with this are given in the reference to Bredon's book.

As indicated before, since we only need the result of the exercise for finite groups, the details of the arguments in the nonfinite case will not be given here.