Remarks on spheres

We begin with a basic decomposition result concerning spheres. Let $\mathrm{Int}\, D^m$ denote the open disk of unit 1 in $\mathbf{R}^m$.

THEOREM. *Suppose that $p, q > 0$. Then the sphere S^{p+q+1} is a union of two open subsets U and V such that U is homeomorphic to $S^p \times \mathrm{Int}\, D^{q+1}$, V is homeomorphic to $\mathrm{Int}\, D^{p+1} \times S^q$, and their intersection is homeomorphic to $S^p \times S^q \times (0,1)$.*

Proof. View S^{p+q+1} as the unit sphere in $\mathbf{R}^{p+q+2} \cong \mathbf{R}^{p+1} \times \mathbf{R}^{q+1}$, and let S_p and S_q denote the unit spheres in $\mathbf{R}^{p+1} \times \{\mathbf{0}\}$ and $\{\mathbf{0}\} \times \mathbf{R}^{q+1}$ respectively. Let $U = S^{p+q+1} - S_q$ and $V = S^{p+q+1} - S_p$. Then there are homeomorphisms

$$h : S^p \times \mathrm{Int}\, D^{q+1} \longrightarrow U \ , \quad k : \mathrm{Int}\, D^{p+1} \times S^q \longrightarrow V$$

defined by the following formulas:

$$h(x,y) \ = \ \left(\sqrt{1 - y^2} \cdot x, \, y \right) \qquad k(x,y) \ = \ \left(x, \, \sqrt{1 - x^2} \cdot y \right)$$

In each case it is a straightforward exercise to write down a formula for the inverse which shows that the inverse is continuous. Also, one has a similar homeomorphism $\varphi : S^p \times (0,1) \times S^q \to S^{p+q+1} - (S_1 \cup S_2) = U \cap V$ which sends (x,t,y) to (tx, sy) where $s = \sqrt{1 - t^2}$.■

SPECIALIZATION TO S^3. In this case $p = q = 1$, and it is instructive to look at the fundamental groups of the various spaces constructed above, for they give an example of a space $X = U \cup V$ such that (i) U and V are open arcwise connected subspaces with an arcwise connected intersection, (ii) the fundamental groups of U and V are nontrivial, (iii) the fundamental group of $X = U \cup V$ is trivial. This is true because the fundamental groups of U and V are infinite cyclic, while the fundamental group of $X = U \cup V$ is trivial.

Here is a more detailed explanation of the situation when $p = q = 1$: The diagram of fundamental groups

$$\pi_1(U) \ \longleftarrow \ \pi_1(U \cap V) \ \longrightarrow \ \pi_1(V)$$

corresponds to the algebraic diagram

$$\mathbf{Z} \longleftarrow \mathbf{Z} \times \mathbf{Z} \longrightarrow \mathbf{Z}$$

where the left and right arrows represent projections onto the first and second factors. Since the generators of $\pi_1(U)$ and $\pi_1(V)$ lift to a pair of free generators for $\pi_1(U \cap V)$, and the respective generators map to the trivial elements in $\pi_1(V)$ and $\pi_1(U)$ respectively, it follows that these generators must map to zero in $\pi_1(X)$, and this in turn yields another proof that $\pi_1(S^3)$ is trivial.■